# 3D Vision

Andreas Geiger, Torsten Sattler

Spring 2017

ETH

# Schedule

| | |
|---|---|
| Feb 20 | Introduction |
| Feb 27 | Geometry, Camera Model, Calibration |
| Mar 6 | Features, Tracking / Matching |
| Mar 13 | **Project Proposals by Students** |
| Mar 20 | Structure from Motion (SfM) + papers |
| Mar 27 | Dense Correspondence (stereo / optical flow) + papers |
| Apr 3 | **Bundle Adjustment & SLAM + papers** |
| Apr 10 | **Student Midterm Presentations** |
| Apr 17 | Easter Break |
| Apr 24 | Multi-View Stereo & Volumetric Modeling + papers |
| May 1 | Labor Day |
| May 8 | 3D Modeling with Depth Sensors + papers |
| May 15 | 3D Scene Understanding + papers |
| May 22 | 4D Video & Dynamic Scenes + papers |
| May 29 | **Student Project Demo Day = Final Presentations** |

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# 3D Vision – Class 6
# Bundle Adjustment and SLAM

- [Triggs, McLauchlan, Hartley, Fitzgibbon, Bundle Adjustment – A Modern Synthesis, Int. Workshop on Vision Algorithms, 1999]
- [Montemerio, Thrun, Koller, Wegbreit, FastSLAM: A Factored Solution to the Simultaneous Localization and Mapping Problem, AAAI 2002]
- Section 2.5 from [Lee, Visual Mapping and Pose Estimation for Self-Driving Cars, PhD Thesis, ETH Zurich, 2014]

Slides : Gim Hee Lee

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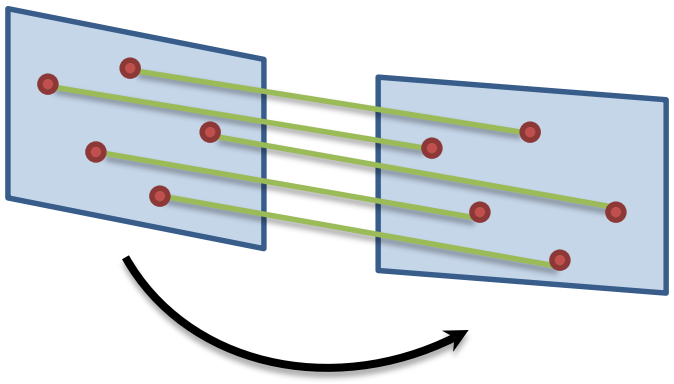
# Lecture Overview

- Bundle Adjustment in Structure-from-Motion
- Simultaneous Localization & Mapping (SLAM)

# Recap: Structure-From-Motion

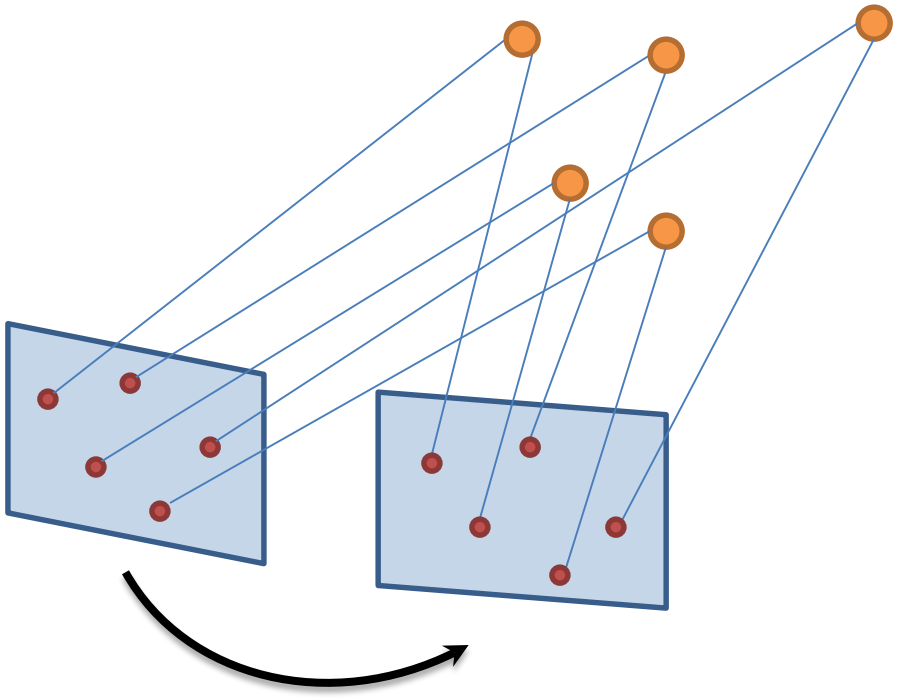
- Two views initialization:
  - 5-Point algorithm (Minimal Solver)
  - 8-Point linear algorithm
  - 7-Point algorithm



E → (R,t)

# Recap: Structure-From-Motion

- Triangulation: 3D Points



E → (R,t)



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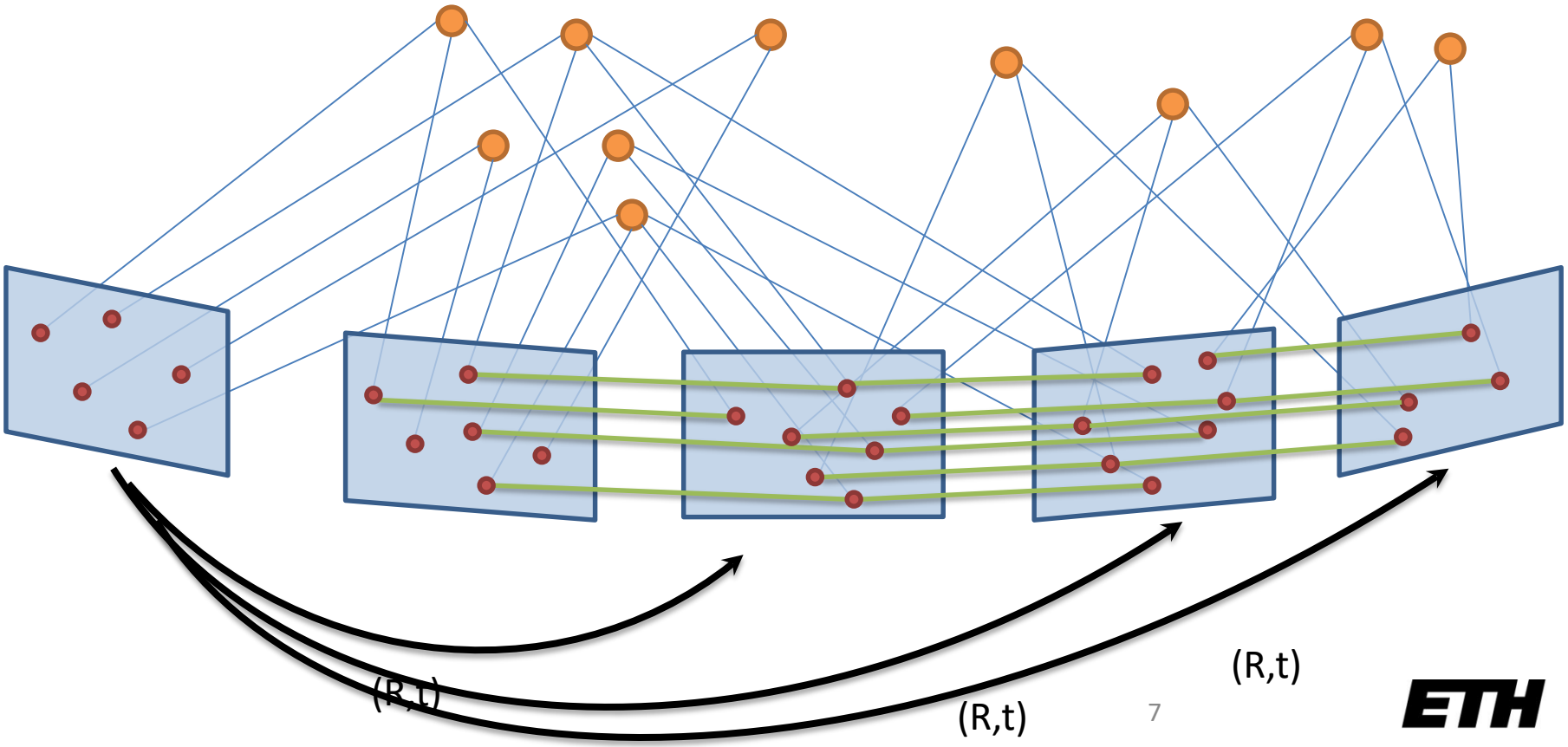
# Recap: Structure-From-Motion

- Subsequent views: Perspective pose estimation



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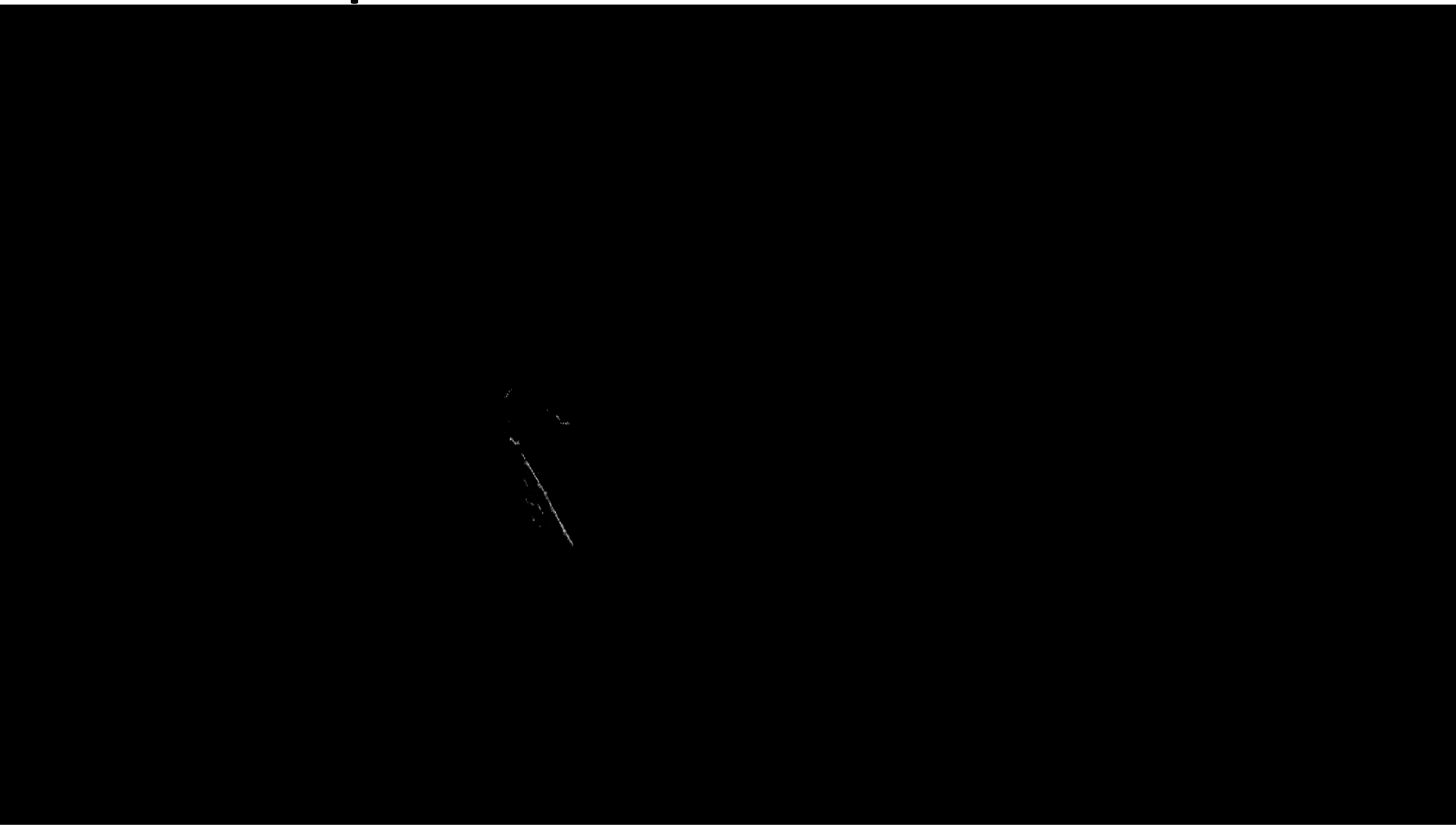
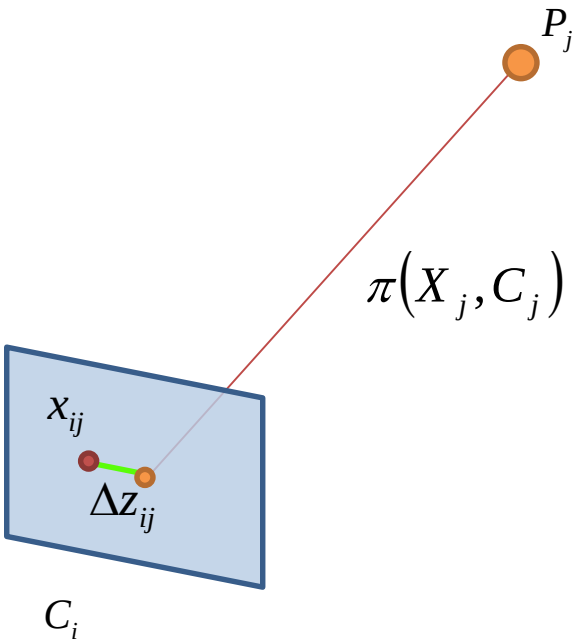
# Recap: Structure-From-Motion



# Bundle Adjustment

- Refinement step in Structure-from-Motion.
- Refine a visual reconstruction to produce jointly optimal 3D structures $P$ and camera poses $C$.
- Minimize total re-projection errors $\Delta z$.




Cost Function:

$$\underset{X}{\operatorname{argmin}} \sum_i \sum_j \left\| \underbrace{x_{ij} - \pi(P_j, C_i)}_{\Delta z_{ij}} \right\|^2_{W_{ij}}$$

$$X = [P, C]$$



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# Bundle Adjustment

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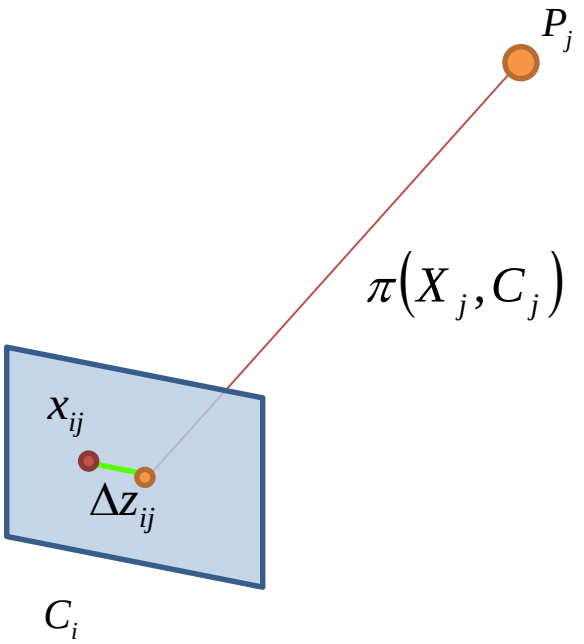


Cost Function:

$$\underset{X}{\operatorname{argmin}} \underbrace{\sum_i \sum_j \Delta z_{ij}^T W_{ij} \Delta z_{ij}}_{f(X)}$$

$W_{ij}^{-1}$ : Measurement error covariance

$X = [P, C]$



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# Bundle Adjustment

- Minimize the cost function: $\underset{X}{\operatorname{argmin}} f(X)$
  1. Gradient Descent
  2. Newton Method
  3. Gauss-Newton
  4. Levenberg-Marquardt

# Bundle Adjustment

1. Gradient Descent

Initialization: $X_k = X_0$

Iterate until convergence

Compute gradient: $g = \left.\frac{\partial f(X)}{\partial X}\right|_{X=X_k} = \Delta Z^T W J$

Update: $X_k \leftarrow X_k - \eta g$

$J = \frac{\partial \pi}{\partial X}$ : Jacobian

$\eta$ : Step size

Slow convergence near minimum point!

# Bundle Adjustment

## 2. Newton Method

2nd order approximation (Quadratic Taylor Expansion):

$$f(X+\delta)\Big|_{X=X_K} \approx f(X) + g\delta + \tfrac{1}{2}\delta^T H \delta\Big|_{X=X_K}$$

Hessian matrix : $H = \dfrac{\partial^2 f(X+\delta)}{\partial \delta^2}\Big|_{X=X_k}$

Find $\delta$ that minimizes $f(X+\delta)\Big|_{X=X_K}$ !

# Bundle Adjustment

2. Newton Method

Differentiate and set to 0 gives:

$$\delta = -H^{-1}g$$

Update: $X_k \leftarrow X_k + \delta$

Computation of H is not trivial and might get stuck at saddle point!

# Bundle Adjustment

3. Gauss-Newton

$$H = J^T W J + \sum_i \sum_j \Delta Z_{ij} W_{ij} \frac{\partial^2 \pi_{ij}}{\partial X^2}$$

$$H \approx J^T W J$$

Normal equation:

$$J^T W J \delta = -J^T W \Delta Z$$

Update: $X_k \leftarrow X_k + \delta$

Might get stuck and slow convergence at saddle point!



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# Bundle Adjustment

## 4. Levenberg-Marquardt

Regularized Gauss-Newton with damping factor $\lambda$ .

$$\underbrace{\left(J^T W J + \lambda I\right)}_{H_{LM}}\delta = -J^T W \Delta Z$$

$\lambda \to 0$ : Gauss-Newton (when convergence is rapid)

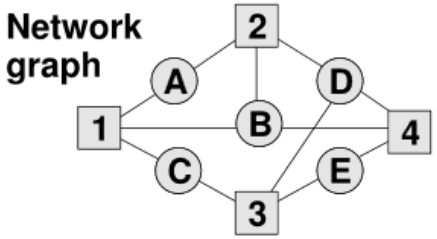
$\lambda \to \infty$ : Gradient descent (when convergence is slow)

Adapt λ during optimization:

- Decrease λ when function value decreases
- Increase λ otherwise

# Structure of the Jacobian and Hessian Matrices

- Sparse matrices since 3D structures are locally observed.




$$J = \begin{array}{c|c|c|c|} & AB\ C\ DE & 1\ 2\ 3\ 4 & K_{12} \\ \hline A1 & \blacksquare & \blacksquare\ \blacksquare & \blacksquare \\ A2 & \blacksquare & \blacksquare\ \blacksquare & \blacksquare \\ \hline B1 & \blacksquare & \blacksquare\ \blacksquare & \blacksquare \\ B2 & \blacksquare & \blacksquare\ \blacksquare & \blacksquare \\ B4 & \blacksquare & \blacksquare\ \blacksquare & \blacksquare \\ \hline C1 & \blacksquare & \blacksquare\ \blacksquare & \blacksquare \\ C3 & \blacksquare & \blacksquare\ \blacksquare & \blacksquare \\ \hline D2 & \blacksquare & \blacksquare\ \blacksquare & \blacksquare \\ D3 & \blacksquare & \blacksquare\ \blacksquare & \blacksquare \\ D4 & \blacksquare & \blacksquare\ \blacksquare & \blacksquare \\ \hline E3 & \blacksquare & \blacksquare\ \blacksquare & \blacksquare \\ E4 & \blacksquare & \blacksquare\ \blacksquare & \blacksquare \\ \hline \end{array}$$



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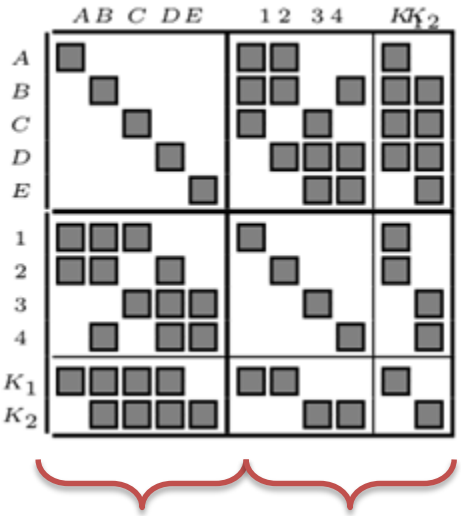
# Efficiently Solving the Normal Equation

- Schur Complement: Exploit structure of H

$$H_{LM}\delta = -J^T W \Delta Z$$

$$H_{LM} =$$



# Efficiently Solving the Normal Equation

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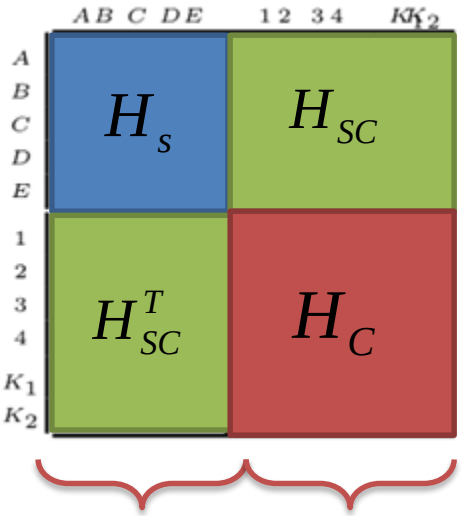




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# Efficiently Solving the Normal Equation

- Schur Complement: Obtain reduced system

$$H_{LM}\delta = -J^T W \Delta Z$$

$$\begin{bmatrix} H_S & H_{SC} \\ H_{SC}^T & H_C \end{bmatrix} \begin{bmatrix} \delta_S \\ \delta_C \end{bmatrix} = \begin{bmatrix} \varepsilon_S \\ \varepsilon_C \end{bmatrix}$$

← 3D Structures

← Camera Parameters

Multiply both sides by: $\begin{bmatrix} I & 0 \\ -H_{SC}^T H_S^{-1} & I \end{bmatrix}$

$$\begin{bmatrix} H_S & H_{SC} \\ 0 & H_C - H_{SC}^T H_S^{-1} H_{SC} \end{bmatrix} \begin{bmatrix} \delta_S \\ \delta_C \end{bmatrix} = \begin{bmatrix} \varepsilon_S \\ \varepsilon_C - \varepsilon_S H_{SC}^T H_S^{-1} \end{bmatrix}$$

# Efficiently Solving the Normal Equation

- Schur Complement: Obtain reduced system

$$\begin{bmatrix} H_S & H_{SC} \\ 0 & H_C - H_{SC}^T H_S^{-1} H_{SC} \end{bmatrix} \begin{bmatrix} \delta_S \\ \delta_C \end{bmatrix} = \begin{bmatrix} \varepsilon_S \\ \varepsilon_C - \varepsilon_S H_{SC}^T H_S^{-1} \end{bmatrix}$$

First solve for $\delta_C$ from:

Easy to invert a block diagonal matrix

$$\underbrace{(H_C - H_{SC}^T H_S^{-1} H_{SC})}_{\text{Schur Complement}} \delta_C = \varepsilon_C - \varepsilon_S H_{SC}^T H_S^{-1}$$

(Sparse and Symmetric Positive Definite Matrix)

Solve for $\delta_{SC}$ by backward substitution.

# Efficiently Solving the Normal Equation

$$(H_C - H_{SC}^T H_S^{-1} H_{SC})\delta_C = \varepsilon_C - \varepsilon_S H_{SC}^T H_S^{-1} \quad \equiv \quad Ax = b$$

Don't solve as $x=A^{-1}b$: A is sparse, but $A^{-1}$ is not!

- Use sparse matrix factorization to solve system
  1. LU Factorization → $A = LU$
  2. QR factorization → $A = QR$
  3. Cholesky Factorization → $A = LL^T$

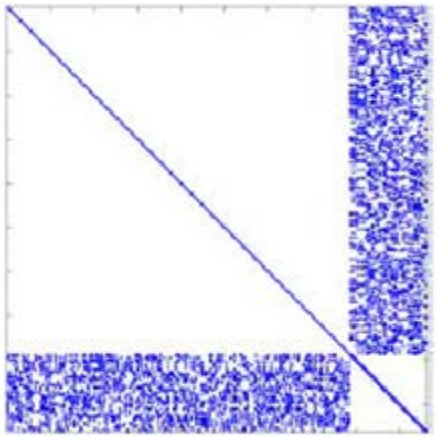
  Solve for x by forward-backward substitutions

- Iterative methods
  1. Conjugate gradient
  2. Gauss-Seidel

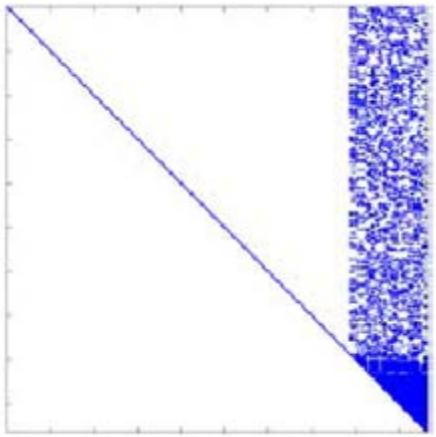
# Efficiently Solving the Normal Equation

- Example for forward-backward substitutions

# Problem of Fill-In



Hessian



Natural Cholesky

# Problem of Fill-In

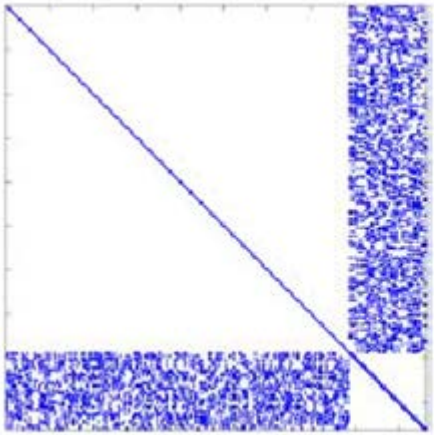
- Reorder sparse matrix to minimize fill-in.

$$\left(P^T AP\right)\left(P^T x\right) = P^T b$$

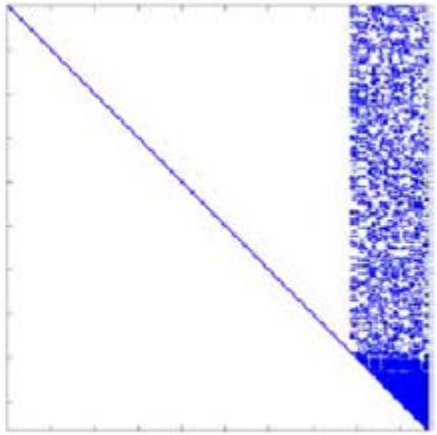
Permutation matrix to reorder $A$

- NP-Complete problem.
- Approximate solutions:
  1. Minimum degree
  2. Column approximate minimum degree permutation
  3. Reverse Cuthill-Mckee.
  4. …

# Problem of Fill-In



Hessian



Natural Cholesky

# Robust Cost Function

- Non-linear least squares: $\underset{X}{\operatorname{argmin}} \sum_{ij} \Delta z_{ij}^T W_{ij} \Delta z_{ij}$
- Maximum log-likelihood solution:

$$-\underset{X}{\operatorname{argmin}} \ln p(Z \mid X)$$

- Assume that:
  1. X is a random variable that follows Gaussian distribution.
  2. All observations are independent.

$$-\underset{X}{\operatorname{argmin}} \ln p(X \mid Z) = -\underset{X}{\operatorname{argmin}} \ln \left\{ \prod_{ij} c_{ij} \exp\left(-\Delta z_{ij}^T W_{ij} \Delta z_{ij}\right) \right\}$$

$$= \underset{X}{\operatorname{argmin}} \sum_{ij} \Delta z_{ij}^T W_{ij} \Delta z_{ij}$$

# Robust Cost Function

- Gaussian distribution assumption is not true in the presence of outliers!
- Causes wrong convergences.

# Robust Cost Function

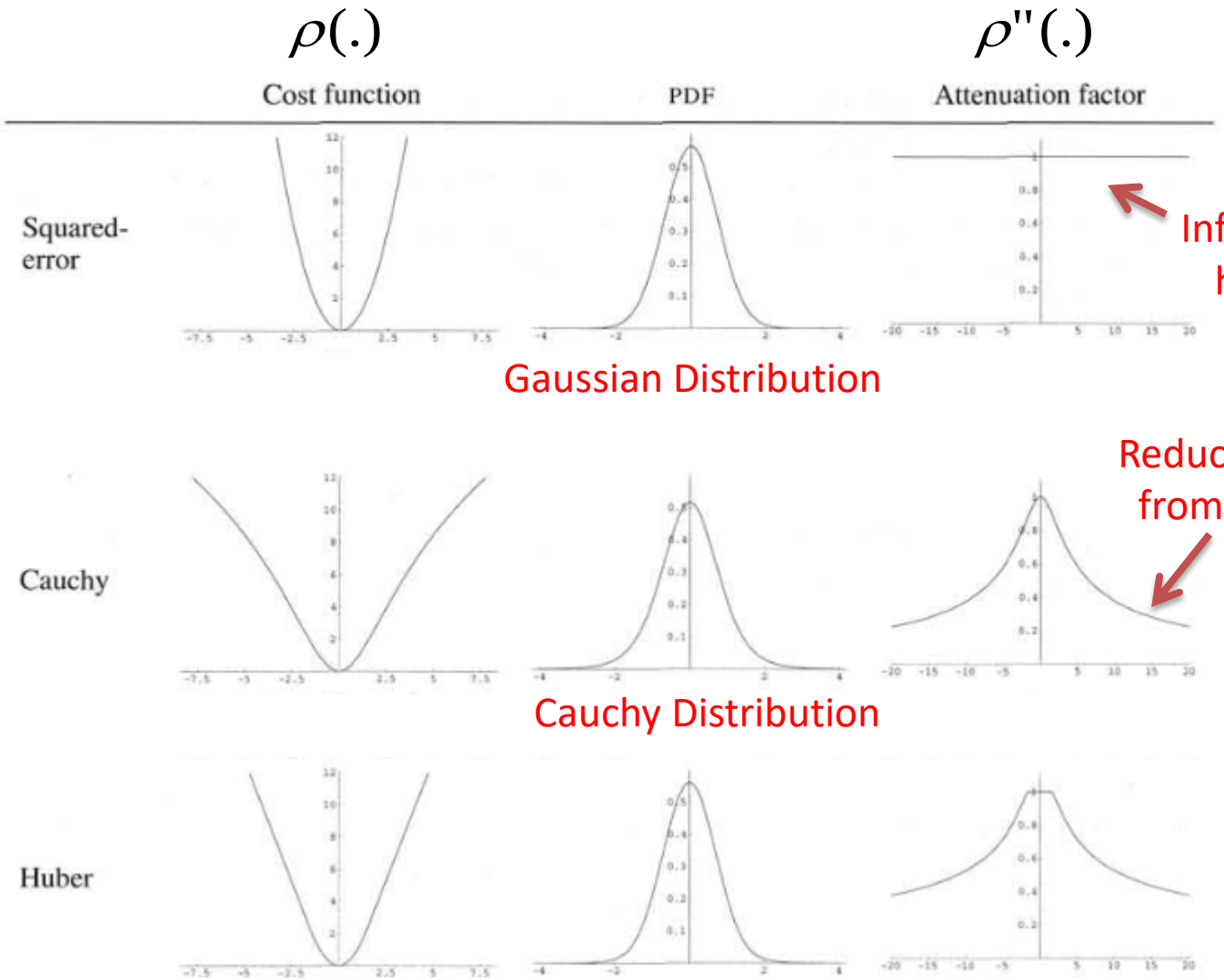
$$\underset{X}{\operatorname{argmin}} \sum_{ij} \rho_{ij}\left(\Delta z_{ij}\right) \equiv \underset{X}{\operatorname{argmin}} \sum_{ij} \Delta z_{ij}^{T} S_{ij} \Delta z_{ij}$$

Robust Cost Function

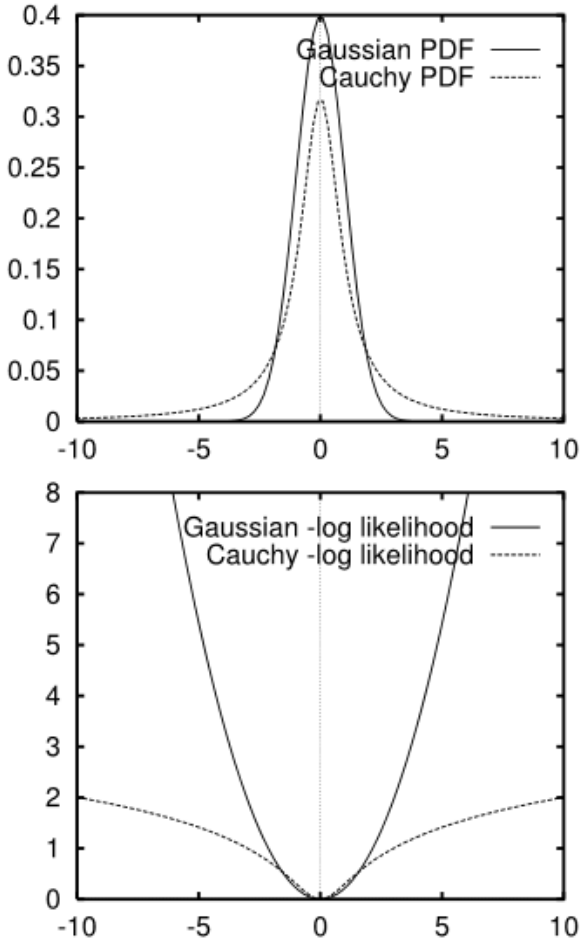
$W_{ij}$ scaled with $\rho''_{ij}$

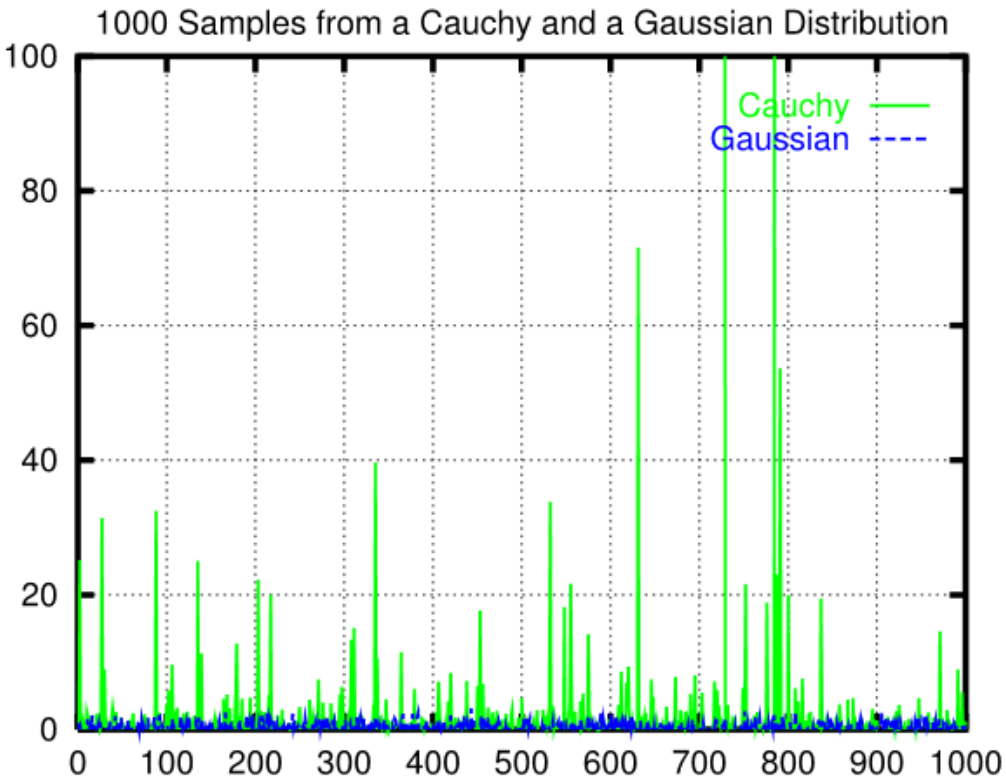
- Similar to iteratively re-weighted least-squares.
- Weight is iteratively rescaled with the attenuating factor $\rho''_{ij}$ .
- Attenuating factor is computed based on current error.

# Robust Cost Function



# Robust Cost Function




Outliers are taken into account in Cauchy!

# State-of-the-Art Solvers

- Google Ceres:
  - https://code.google.com/p/ceres-solver/
- g2o:
  - https://openslam.org/g2o.html
- GTSAM:
  - https://collab.cc.gatech.edu/borg/gtsam/
- Multicore Bundle Adjustment
  - http://grail.cs.washington.edu/projects/mcba/

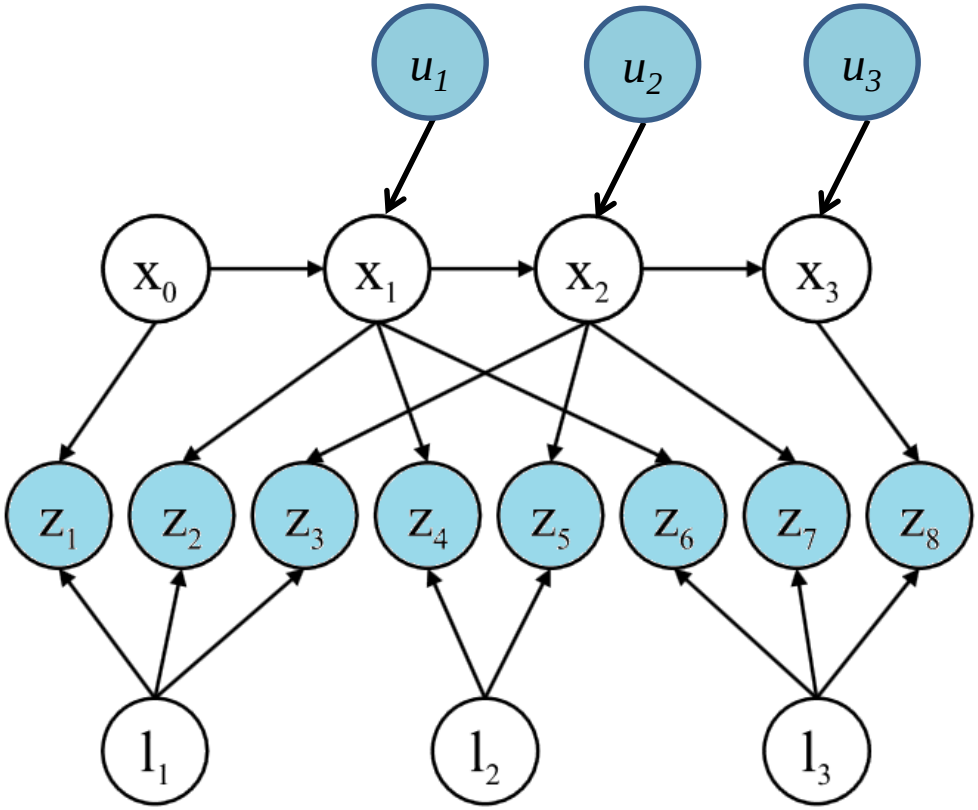
# Lecture Overview

- Bundle Adjustment in Structure-from-Motion
- Simultaneous Localization & Mapping (SLAM)

# Simultaneous Localization & Mapping (SLAM)

- Robot navigates in unknown environment:
  - Estimate its own pose
  - Acquire a map model of its environment.
- Chicken-and-Egg problem:
  - Map is needed for localization (pose estimation).
  - Pose is needed for mapping.
- Highly related to Structure-From-Motion.

# Full SLAM: Problem Definition




Control actions

Robot poses

Observations

Map landmarks



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# Full SLAM: Problem Definition

- Maximum a Posteriori (MAP) solution:

$$\underset{X,L}{\operatorname{argmax}}\, p(X, L \mid Z, U) = \underset{X,L}{\operatorname{argmax}}\, p(X_0) \prod_{i=1}^{M} p(x_i \mid x_{i-1}, u_i) \prod_{k=1}^{K} p(z_k \mid x_{ik}, l_{jk})$$

# Full SLAM

$$\underset{X,L}{\operatorname{argmax}}\, p(X, L \mid Z, U) = \underset{X,L}{\operatorname{argmax}}\, p(X_0) \prod_{i=1}^{M} p(x_i \mid x_{i-1}, u_i) \prod_{k=1}^{K} p(z_k \mid x_{ik}, l_{jk})$$

Negative log-likelihood →

$$= -\underset{X,L}{\operatorname{argmin}} \left\{ \sum_{i=1}^{M} \ln p(x_i \mid x_{i-1}, u_i) + \sum_{k=1}^{K} \ln p(z_k \mid x_{ik}, l_{jk}) \right\}$$

Likelihoods:

$$p(x_i \mid x_{i-1}, u_i) \propto \exp\{-\| f(x_{i-1}, u) - x_i \|^2_{\Lambda_i}\}$$

Process model

$$p(z_k \mid x_{ik}, l_{jk}) \propto \exp\{-\| h(x_{ik}, l_{jk}) - z_k \|^2_{\Sigma_k}\}$$

Measurement model

# Full SLAM

$$\underset{X,L}{\operatorname{argmax}}\, p(X, L \mid Z, U) = -\underset{X,L}{\operatorname{argmin}} \left\{ \sum_{i=1}^{M} \ln p(x_i \mid x_{i-1}, u_i) + \sum_{k=1}^{K} \ln p(z_k \mid x_{ik}, l_{jk}) \right\}$$

Putting the likelihoods into the equation:

$$\underset{X,L}{\operatorname{argmax}}\, p(X, L \mid Z, U) = \underset{X,L}{\operatorname{argmin}} \left\{ \sum_{i=1}^{M} \left\| f(x_{i-1}, u_i) - x_i \right\|_{\Lambda_i}^2 + \sum_{k=1}^{K} \left\| h(x_{ik}, l_{jk}) - z_k \right\|_{\Sigma_k}^2 \right\}$$

Minimization can be done with Levenberg-Marquardt (similar to bundle adjustment problem)!

# Full SLAM

Normal Equations:

Weight made up of $\Lambda_i$ , $\Sigma_k$

$$\left(J^T W J + \lambda I\right)\delta = -J^T W \Delta Z$$

Jacobian made up of $\frac{\partial f}{\partial x}$ , $\frac{\partial f}{\partial u}$ , $\frac{\partial h}{\partial x}$ , $\frac{\partial h}{\partial l}$

Can be solved with sparse matrix factorization or iterative methods

Solving the full SLAM problem rather expensive for larger scenes

# Online SLAM: Problem Definition

- Estimate current pose $x_t$ and full map $L$:

$$p(x_t, L \mid Z, U) = \underbrace{\int\int \mathrm{K} \int p(X, L \mid Z, U)\, dx_1\, dx_2 \ldots dx_{t-1}}$$

Previous poses are marginalized out

- Inference with:
  1. (Extended) Kalman Filter (EKF SLAM)
  2. Particle Filter (FastSLAM)

# EKF SLAM

- Assumes: pose $x_t$ and map $L$ are random variables that follow Gaussian distributions.
- Hence,

$$p(x_t, L \mid Z, U) \sim \mathrm{N}(\mu, \Sigma)$$

Mean — Error covariance

- (Extended) Kalman Filter iteratively
  - Predicts pose & map based on process model
  - Corrects prediction based on observations

# EKF SLAM

Prediction:

$$\bar{\mu}_t = f(u_t, \mu_{t-1})$$ ← Process model

$$\bar{\Sigma}_t = F_t \Sigma_{t-1} F_t^T + R_t$$ ← Error propagation with process noise

Correction:

$$y_t = z_t - h(\bar{\mu}_t)$$ ← Measurement residual (innovation)

$$K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1}$$ ← Kalman gain

$$\mu_t = \bar{\mu}_t + K_t y_t$$ ← Update mean

$$\Sigma_t = (I - K_t H_t)\bar{\Sigma}_t$$ ← Update covariance

Measurement Jacobian $H_t = \frac{\partial h(\bar{\mu}_t)}{\partial x_t}$ Process Jacobian $F_t = \frac{\partial f(u_t, \mu_{t-1})}{\partial x_{t-1}}$

# Structure of Mean and Covariance

$$\mu_t = \begin{pmatrix} x \\ y \\ \theta \\ l_1 \\ l_2 \\ \mathrm{M} \\ l_N \end{pmatrix}, \quad \Sigma_t = \begin{pmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{x\theta} & \sigma_{xl_1} & \sigma_{xl_2} & \mathrm{L} & \sigma_{xl_N} \\ \sigma_{xy} & \sigma_y^2 & \sigma_{y\theta} & \sigma_{yl_1} & \sigma_{yl_2} & \mathrm{L} & \sigma_{yl_N} \\ \sigma_{x\theta} & \sigma_{y\theta} & \sigma_\theta^2 & \sigma_{\theta l_1} & \sigma_{\theta l_2} & \mathrm{L} & \sigma_{\theta l_N} \\ \sigma_{xl_1} & \sigma_{yl_1} & \sigma_{\theta l_1} & \sigma_{l_1}^2 & \sigma_{l_1 l_2} & \mathrm{L} & \sigma_{l_1 l_N} \\ \sigma_{xl_2} & \sigma_{yl_2} & \sigma_{\theta l_2} & \sigma_{l_1 l_2} & \sigma_{l_2}^2 & \mathrm{L} & \sigma_{l_2 l_N} \\ \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{O} & \mathrm{M} \\ \sigma_{xl_N} & \sigma_{yl_N} & \sigma_{\theta l_N} & \sigma_{l_1 l_N} & \sigma_{l_2 l_N} & \mathrm{L} & \sigma_{l_N}^2 \end{pmatrix}$$

Covariance is a dense matrix that grows with increasing map features!

True robot and map states might not follow unimodal Gaussian distribution!

# Particle Filtering: FastSLAM

- Particles represents samples from the posterior distribution $p(x_t, L \mid Z, U)$ .
- $p(x_t, L \mid Z, U)$ can be any distribution (need not be Gaussian).

# FastSLAM

Each particle represents:

$$p_t^m = \{x_t^m, < \mu_{1,t}^m, \Sigma_{1,t}^m >, < \mu_{2,t}^m, \Sigma_{2,t}^m > \ldots < \mu_{N,t}^m, \Sigma_{N,t}^m >\}$$

Robot state

Landmark state
(mean and covariance)

$$x_t^m \sim p(x_t \mid x_{t-1}, u_t)$$ ← Sample the robot state from the process model

$$p(L_{n,t}^m \mid x_t^m, z_t)$$ ← N Kalman filter Landmark updates

$$w_t^m \propto p(z_t \mid L_t^m, x_t^m)$$ ← Weight update

Resampling based on current state



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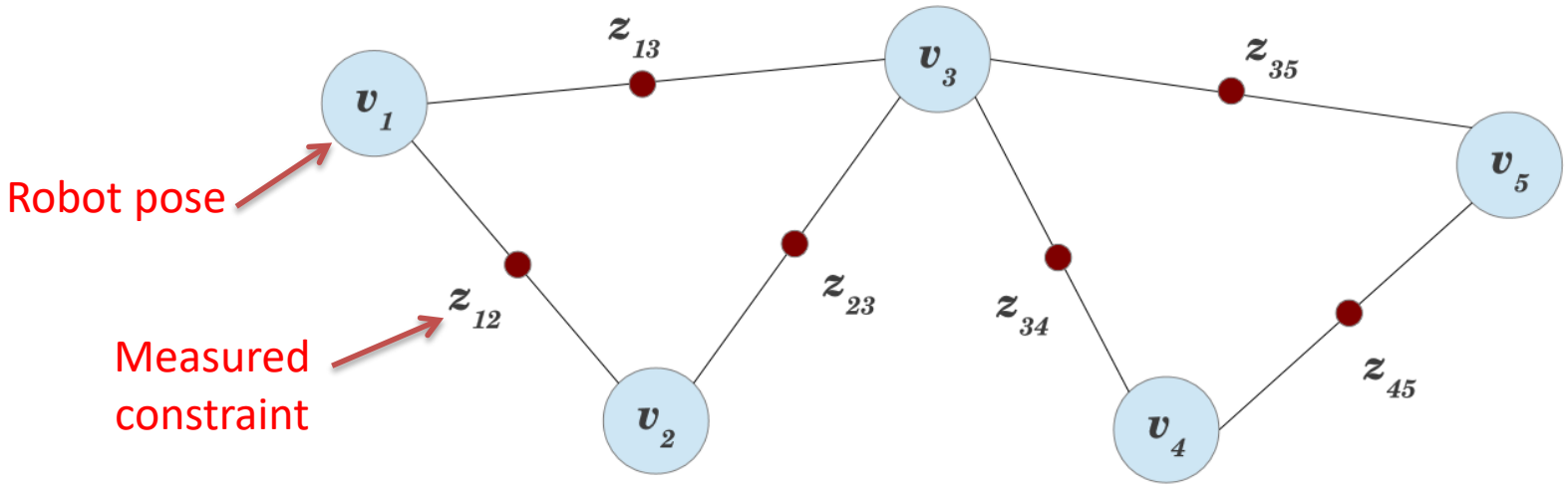
# FastSLAM

- Many particles needed for accurate results.
- Computationally expensive for high state dimensions.

# Pose-Graph SLAM




- Constraints: Relative pose estimates from 3D structure.
- Don't update 3D structure (fixed wrt. to some pose).
- Optimizes poses as $\underset{X}{\operatorname{argmin}} \sum_{ij} \left\| z_{ij} - h(v_i, v_j) \right\|^2_{\Sigma_{ij}}$

Relative transformation between poses

- Can be used to minimize loop-closure errors.



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# Summary

- Bundle Adjustment
  - Refine 3D points and poses in Structure-From-Motion.
  - Efficient computation by exploiting structure & sparsity.
  - Core step in every Structure-From-Motion (SFM) pipeline.
- Simultaneous Localization and Mapping
  - Very similar to Incremental SFM.
  - Typically includes some motion model.
  - Two general approaches to SLAM:
    - (Local) Bundle Adjustment (not discussed in lecture)
    - Filter-based techniques (EKF SLAM, FastSLAM)
  - Pose-Graph SLAM (loop-closure handling)

# Schedule

| | |
|---|---|
| Feb 20 | Introduction |
| Feb 27 | Geometry, Camera Model, Calibration |
| Mar 6 | Features, Tracking / Matching |
| Mar 13 | **Project Proposals by Students** |
| Mar 20 | Structure from Motion (SfM) + papers |
| Mar 27 | Dense Correspondence (stereo / optical flow) + papers |
| Apr 3 | Bundle Adjustment & SLAM + papers |
| Apr 10 | **Student Midterm Presentations** |
| Apr 17 | Easter Break |
| Apr 24 | Multi-View Stereo & Volumetric Modeling + papers |
| May 1 | Labor Day |
| May 8 | 3D Modeling with Depth Sensors + papers |
| May 15 | 3D Scene Understanding + papers |
| May 22 | 4D Video & Dynamic Scenes + papers |
| May 29 | **Student Project Demo Day = Final Presentations** |

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# Next week: Midterm Presentations

**Reminder: Prepare short presentation for Monday!**