3D Vision

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Spring 2024

ETH

Schedule

Feb 19	Introduction
Feb 26	**Geometry, Camera Model, Calibration**
Mar 4	Guest lecture + Features, Tracking / Matching
Mar 11	**Project Proposals by Students**
Mar 18	3DV conference
Mar 25	Structure from Motion (SfM) + papers
Apr 1	Easter break
Apr 8	Dense Correspondence (stereo / optical flow) + papers
Apr 15	Bundle Adjustment & SLAM + papers
Apr 22	**Student Midterm Presentations**
Apr 29	Multi-View Stereo & Volumetric Modeling + papers
May 6	3D Modeling with Depth Sensors + papers
May 13	Guest lecture + papers
May 20	Holiday

3D Vision– Class 2

Projective Geometry and Camera Model

points, lines, planes, conics and quadrics
Transformations, camera model

Chapters 1, 2 and 5 in Hartley and Zisserman 1st edition
Or Chapters 2, 3 and 6 in 2nd edition
See also Chapter 2 in Szeliski book

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Topics Today

- Lecture intended as a review of material covered in Computer Vision lecture
- Probably the hardest lecture (since very theoretic) in the class …
- … but fundamental for any type of 3D Vision application
- Key takeaways:
  - 2D primitives (points, lines, conics) and their transformations
  - 3D primitives and their transformations
  - Camera model and camera calibration

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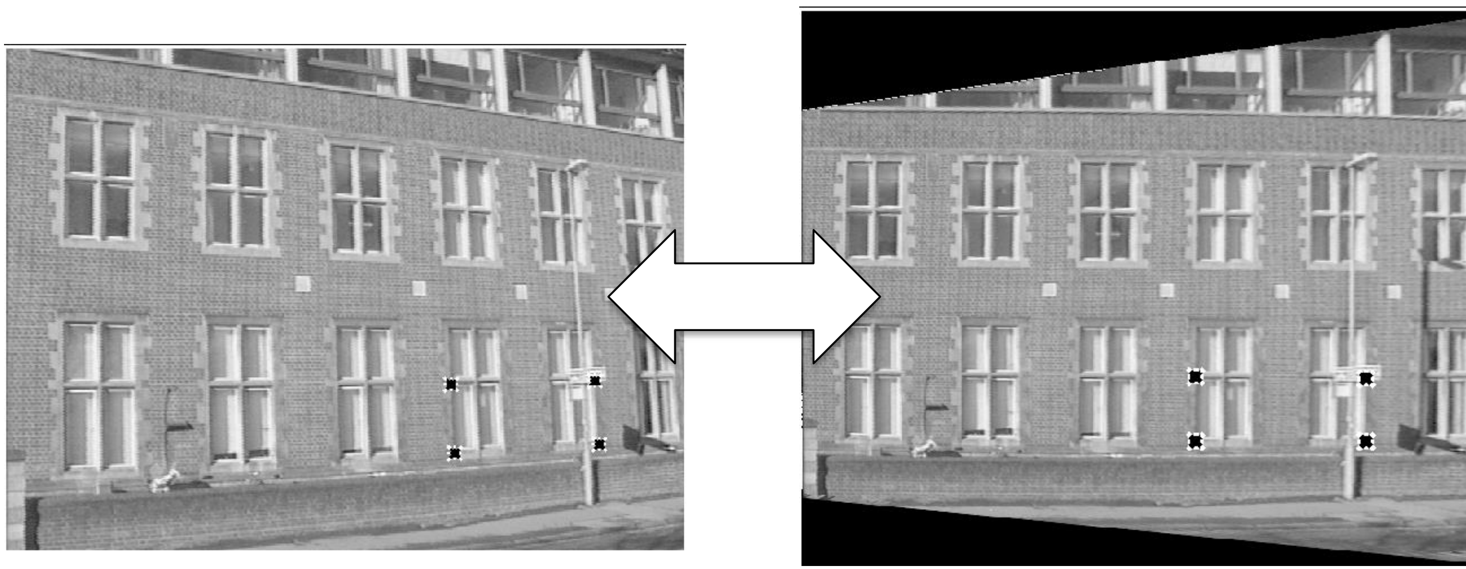
Overview

- **2D Projective Geometry**
- 3D Projective Geometry
- Camera Models & Calibration

2D Projective Geometry?

Projections of planar surfaces



A. Criminisi. *Accurate Visual Metrology from Single and Multiple Uncalibrated Images*. PhD Thesis 1999.

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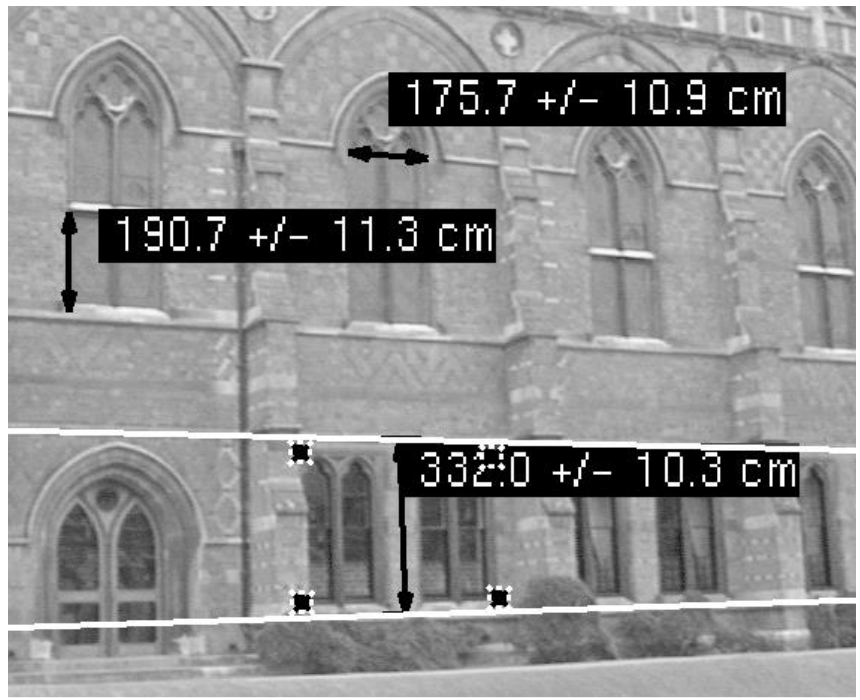
2D Projective Geometry?

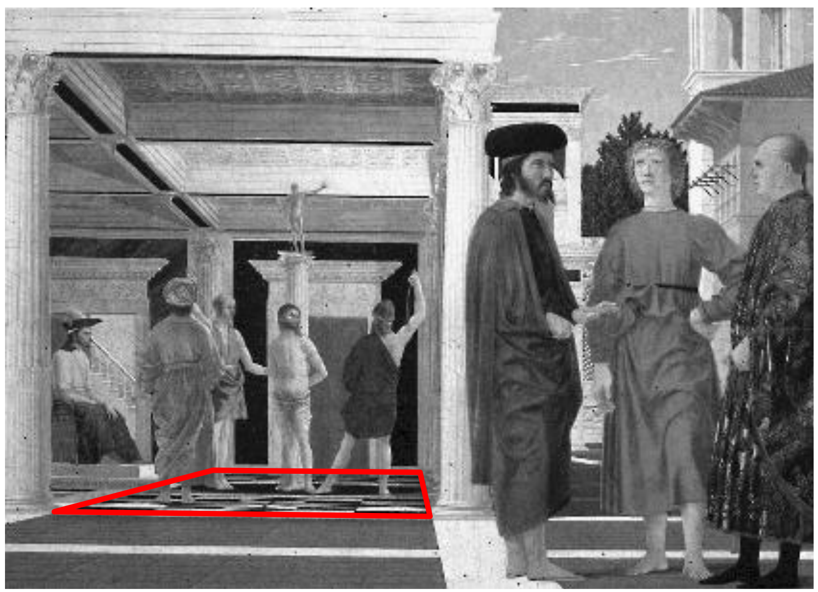
Measure distances



A. Criminisi. *Accurate Visual Metrology from Single and Multiple Uncalibrated Images*. PhD Thesis 1999.

2D Projective Geometry?

Discovering details



Piero della Francesca, La Flagellazione di Cristo (1460)

A. Criminisi. *Accurate Visual Metrology from Single and Multiple Uncalibrated Images*. PhD Thesis 1999.

2D Projective Geometry?

Image Stitching

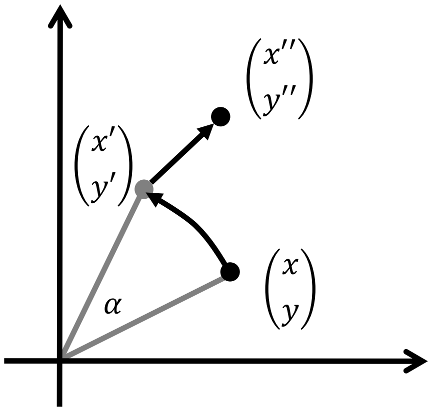
2D Projective Geometry?

Image Stitching

2D Euclidean Transformations

- Rotation (around origin)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



- Translation

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

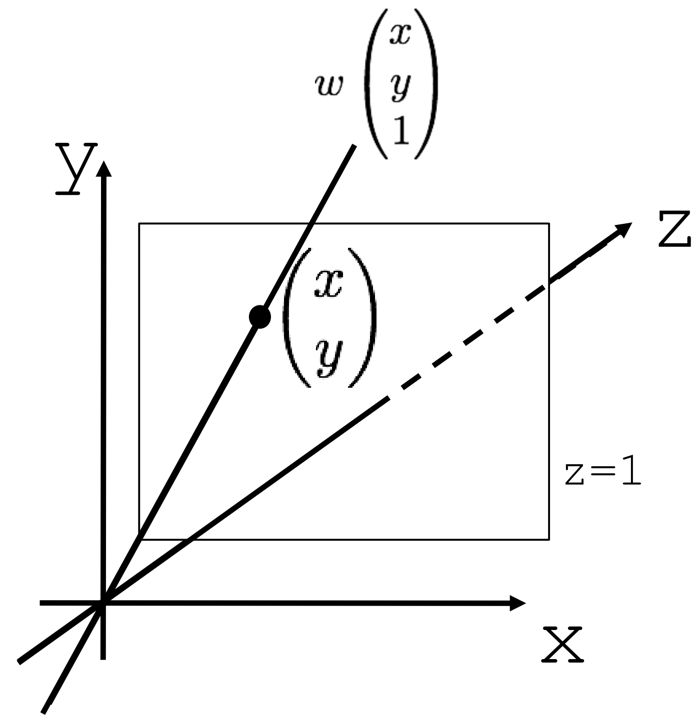
- "Extended coordinates"

$$\begin{pmatrix} x'' \\ y'' \\ 1 \end{pmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & t_x \\ \sin\alpha & \cos\alpha & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Homogeneous Coordinates

Homogenous coordinates

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow w \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \quad , \quad w \neq 0$$

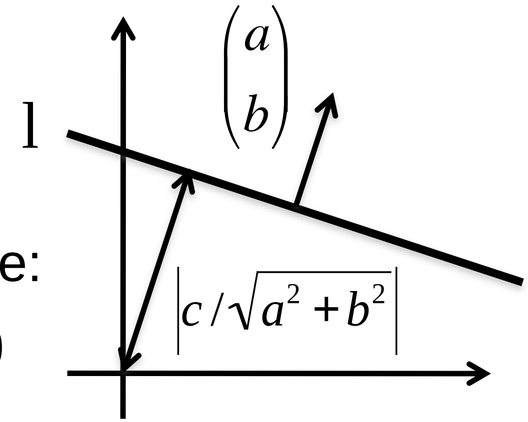


Equivalence class of vectors

$$\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -9 \\ 6 \\ -3 \end{pmatrix}$$

2D projective space: $\mathbb{P}^2 = \mathbb{R}^3 \setminus \{(0,0,0)\}$

Homogeneous Coordinates



(Homogeneous) representation of 2D line:

$$ax + by + c = 0 \qquad (a,b,c)^{\mathsf{T}}(x,y,1) = 0$$

The point x lies on the line l if and only if $\mathrm{l}^{\mathsf{T}}\mathrm{x} = 0$

Note that scale is unimportant for incidence relation

$$(a,b,c)^{\mathsf{T}} \sim k(a,b,c)^{\mathsf{T}}, \forall k \neq 0 \qquad (x,y,1)^{\mathsf{T}} \sim k(x,y,1)^{\mathsf{T}}, \forall k \neq 0$$

Homogeneous coordinates $(x_1, x_2, x_3)^{\mathsf{T}}$ but only 2DOF

Inhomogeneous coordinates $(x,y)^{\mathsf{T}} = (x_1/x_3,\, x_2/x_3)^{\mathsf{T}}$

2D Projective Transformations

Definition:

A *projectivity* is an invertible mapping h from $\mathbb{P}^2$ to itself such that three points x_1, x_2, x_3 lie on the same line if and only if $h(x_1), h(x_2), h(x_3)$ do.

Theorem:

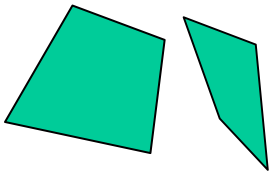
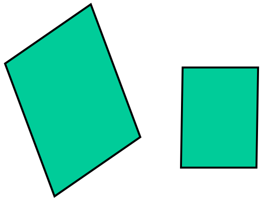
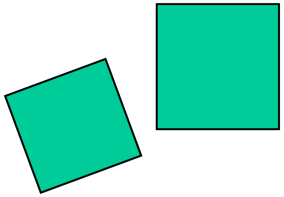
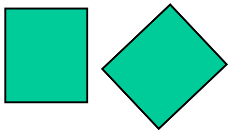
A mapping $h : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ is a **projectivity** if and only if there exist a non-singular 3x3 matrix $\mathbf{H}$ such that for any point in P^2 represented by a vector x it is true that $h(\mathrm{x})=\mathbf{H}\mathrm{x}$

Definition: Projective transformation

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{or} \quad \mathrm{x}' = \mathbf{H}\,\mathrm{x} \quad \text{8DOF}$$

projectivity = collineation = proj. transformation = **homography**

Hierarchy of 2D Transformations

		transformed squares	invariants
Projective 8dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, order of contact (intersection, tangency, inflection, etc.), cross ratio
Affine 6dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on parallel lines (e.g. midpoints), linear combinations of vectors (centroids), **The line at infinity $\mathbf{l}_\infty$**
Similarity 4dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratios of lengths, angles, The circular points I,J
Euclidean 3dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Absolute lengths, angles, areas

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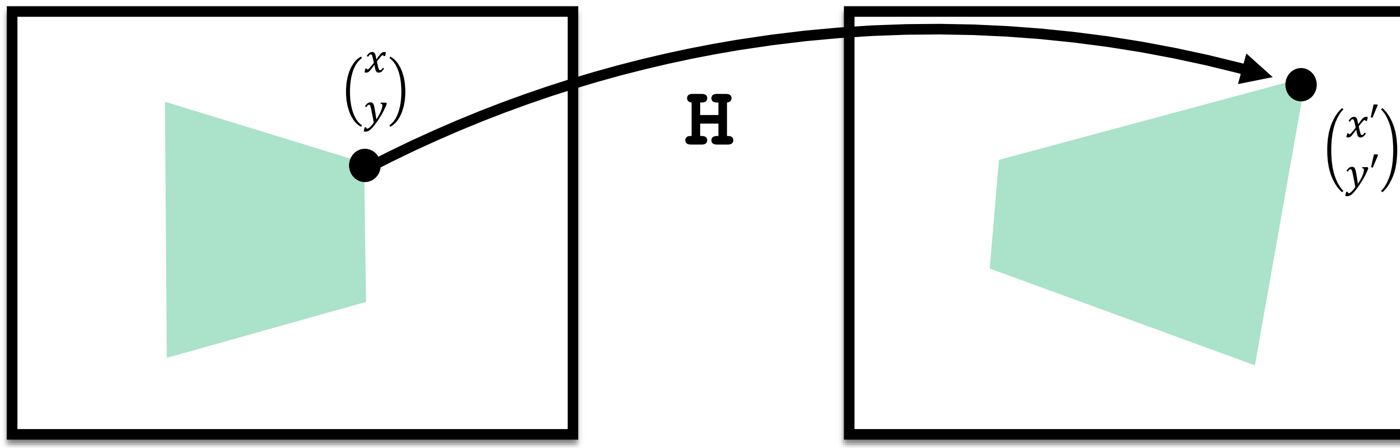
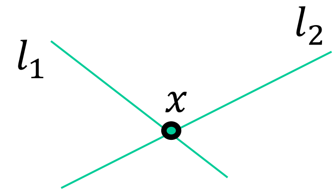
Working with Homogeneous Coordinates



- "Homogenize": $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$
- Apply **H**: $\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \mathtt{H} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$
- De-homogenize: $\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} \mapsto \begin{pmatrix} x''/z'' \\ y''/z'' \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$

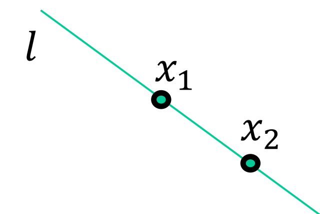
Lines to Points, Points to Lines

- Intersections of lines



Find x such that $\begin{cases} l_1^T x = 0 \\ l_2^T x = 0 \end{cases}$ $\quad x = l_1 \times l_2$

- Line through two points

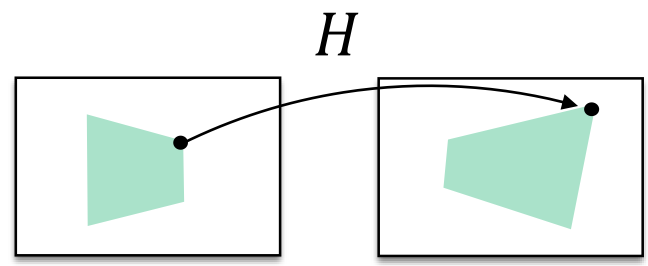


Find l such that $\begin{cases} l^T x_1 = 0 \\ l^T x_2 = 0 \end{cases}$ $\quad l = x_1 \times x_2$

Transformation of Points and Lines

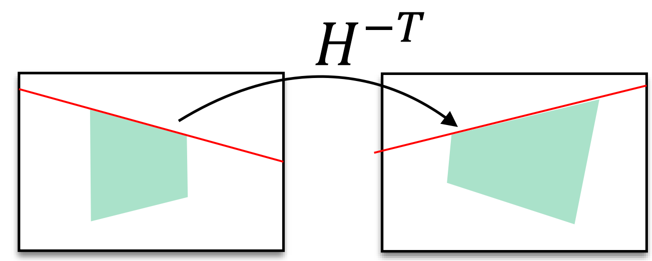
- For a point transformation

$$x' = Hx$$



- Transformation for lines

$$l' = H^{-T} l$$



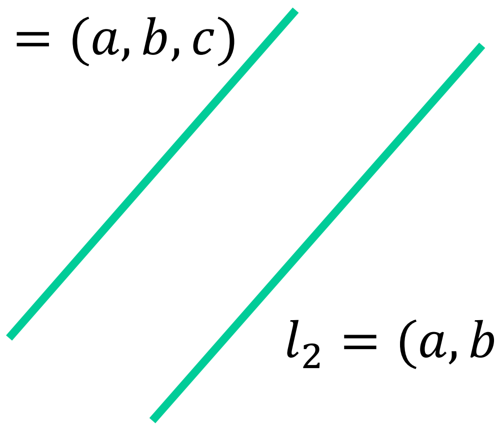
$$l^T x = 0 \Rightarrow l^T (H^{-1} H) x = 0 \Rightarrow \underbrace{(H^{-T} l)^T}_{l'} \underbrace{Hx}_{x'} = 0$$

Ideal Points

- Intersections of parallel lines?

$$l_1 \times l_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} a \\ b \\ c' \end{pmatrix} = (c' - c) \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix}$$

$l_1 = (a, b, c)$

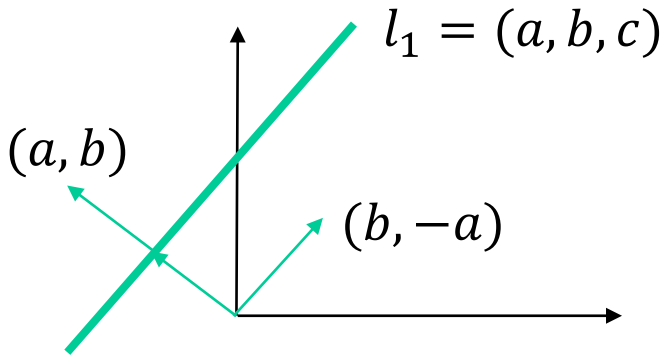


$l_2 = (a, b, c')$

- Parallel lines intersect in *Ideal Points* $\left(x_1, x_2, 0\right)^{\mathrm{T}}$

Ideal Points

- Ideal points correspond to *directions*



Ideal point $\begin{pmatrix} b \\ -a \\ 0 \end{pmatrix}$

- Unaffected by translation

$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} r_{11}x + r_{12}y \\ r_{21}x + r_{22}y \\ 0 \end{pmatrix}$$

The Line at Infinity

- Line through two ideal points?

$$\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \times \begin{pmatrix} x' \\ y' \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ xy' - x'y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = l_\infty$$

- Line at infinity $\mathrm{l}_\infty = (0,0,1)^\mathsf{T}$ intersects all ideal points

$$l_\infty^T x = l_\infty^T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 = 0$$

$$\mathbb{P}^2 = \mathbb{R}^2 \cup l_\infty$$

Note that in $\mathbb{P}^2$ there is no distinction between ideal points and others

The Line at Infinity

The line at infinity $l_\infty=(0,0,1)^\mathsf{T}$ is a fixed line under a projective transformation H if and only if H is an affinity (affine transformation)

$$l'_\infty = \mathbf{H}_A^{-\mathsf{T}}\, l_\infty = \begin{bmatrix} \mathbf{A}^{-\mathsf{T}} & 0 \\ -t^\mathsf{T}\mathbf{A}^{-\mathsf{T}} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = l_\infty$$

Note: not fixed pointwise

Affine trans.

$$\boldsymbol{H}_A = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{t} \\ \boldsymbol{0}^{\boldsymbol{T}} & 1 \end{bmatrix}$$

Conics

- Curve described by 2nd-degree equation in the plane

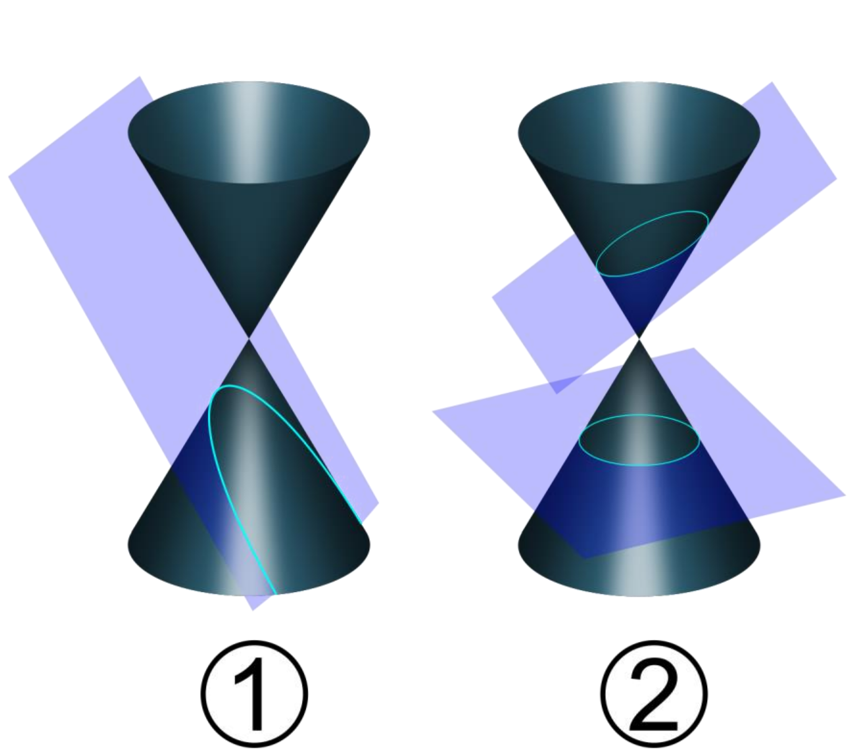
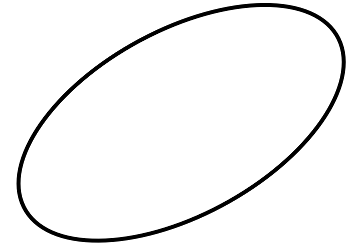


Image source: Wikipedia

Conics



- Curve described by 2nd-degree equation in the plane

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

or *homogenized* $x \mapsto x_1/x_3,\ y \mapsto x_2/x_3$

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

or in matrix form $\mathbf{x}^T C \mathbf{x} = 0$

$$(x_1\ x_2\ x_3)\begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

- 5DOF (degrees of freedom): $\{a:b:c:d:e:f\}$ (defined up to scale)

Five Points Define a Conic

For each point the conic passes through

$$ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f = 0$$

or

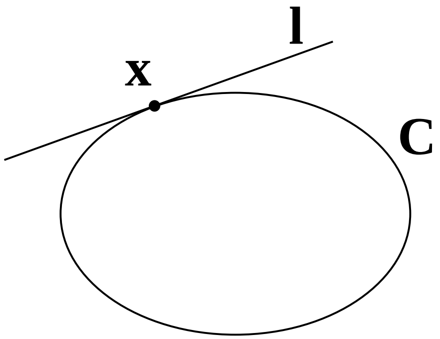
$$\left(x_i^2, x_iy_i, y_i^2, x_i, y_i, 1\right)\mathbf{c} = 0 \qquad \mathbf{c} = \left(a,b,c,d,e,f\right)^{\mathsf{T}}$$

stacking constraints yields

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix}\mathbf{c} = 0$$

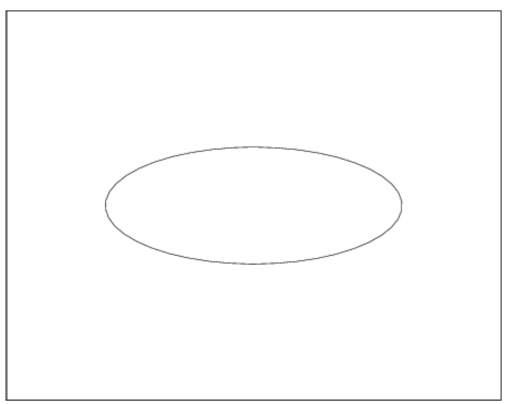
Tangent Lines to Conics

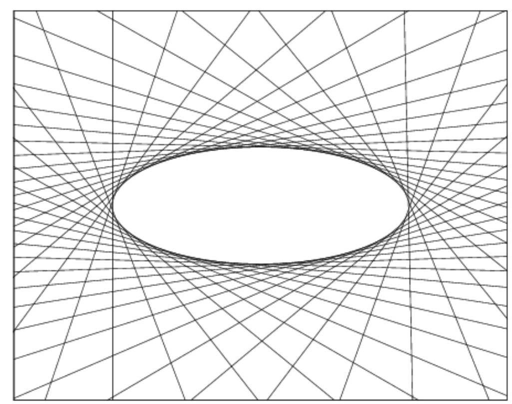
The line **l** tangent to **C** at point **x** on **C** is given by **l**=**Cx**



Dual Conics

- A line tangent to the conic $\mathbf{C}$ satisfies $\mathbf{l}^{\mathsf{T}} \mathbf{C}^{*} \mathbf{l} = 0$
- In general ($\mathbf{C}$ full rank): $\mathbf{C}^{*} = \mathbf{C}^{-1}$
- Dual conics = line conics = conic envelopes

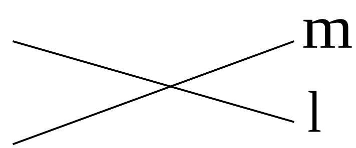




Degenerate Conics

- A conic is degenerate if matrix $\mathbf{C}$ is not of full rank

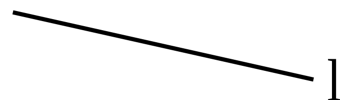
e.g. two lines (rank 2)



$$\mathbf{C} = \mathbf{l}\mathbf{m}^\mathsf{T} + \mathbf{m}\mathbf{l}^\mathsf{T}$$

e.g. repeated line (rank 1)

$$\mathbf{C} = \mathbf{l}\mathbf{l}^\mathsf{T}$$



- Degenerate line conics: 2 points (rank 2), double point (rank1)
- Note that for degenerate conics $\left(\mathbf{C}^*\right)^* \neq \mathbf{C}$

Transformation of Points, Lines and Conics

- For a point transformation

$$x' = Hx$$

- Transformation for lines

$$l' = H^{-T} l$$

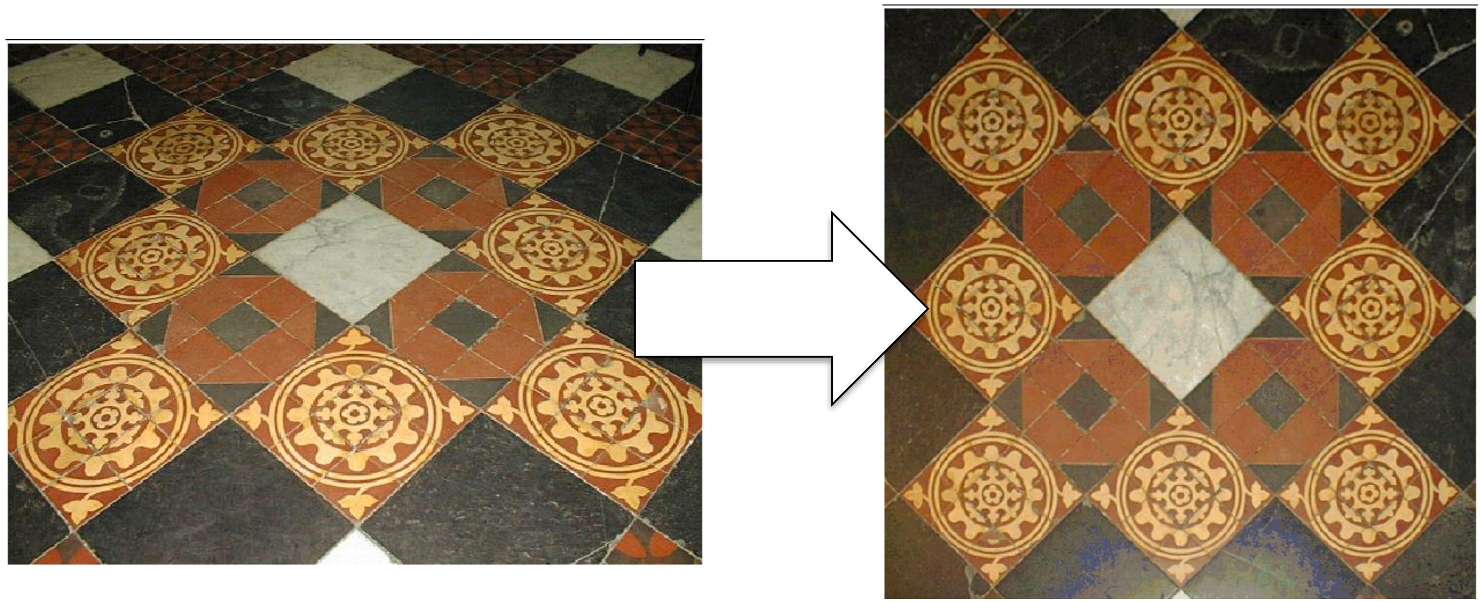
- Transformation for conics

$$C' = H^{-T} C H^{-1}$$

- Transformation for dual conics

$$C^{*\prime} = H C^{*} H^{T}$$

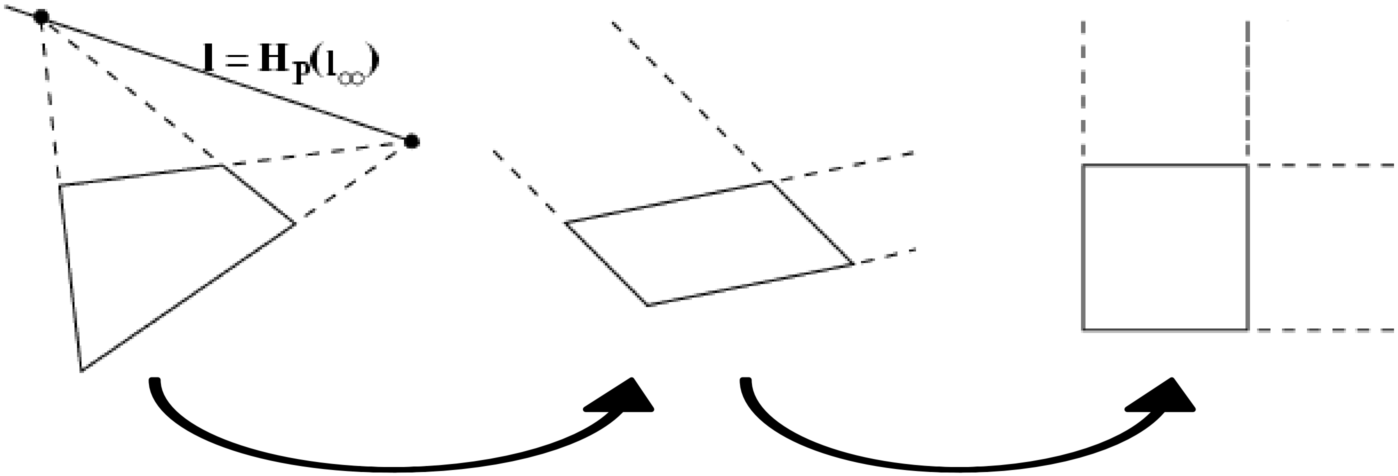
Application: Removing Perspective



Two stages:

- From perspective to affine transformation via the line at infinitiy
- From affine to similarity transformation via the circular points

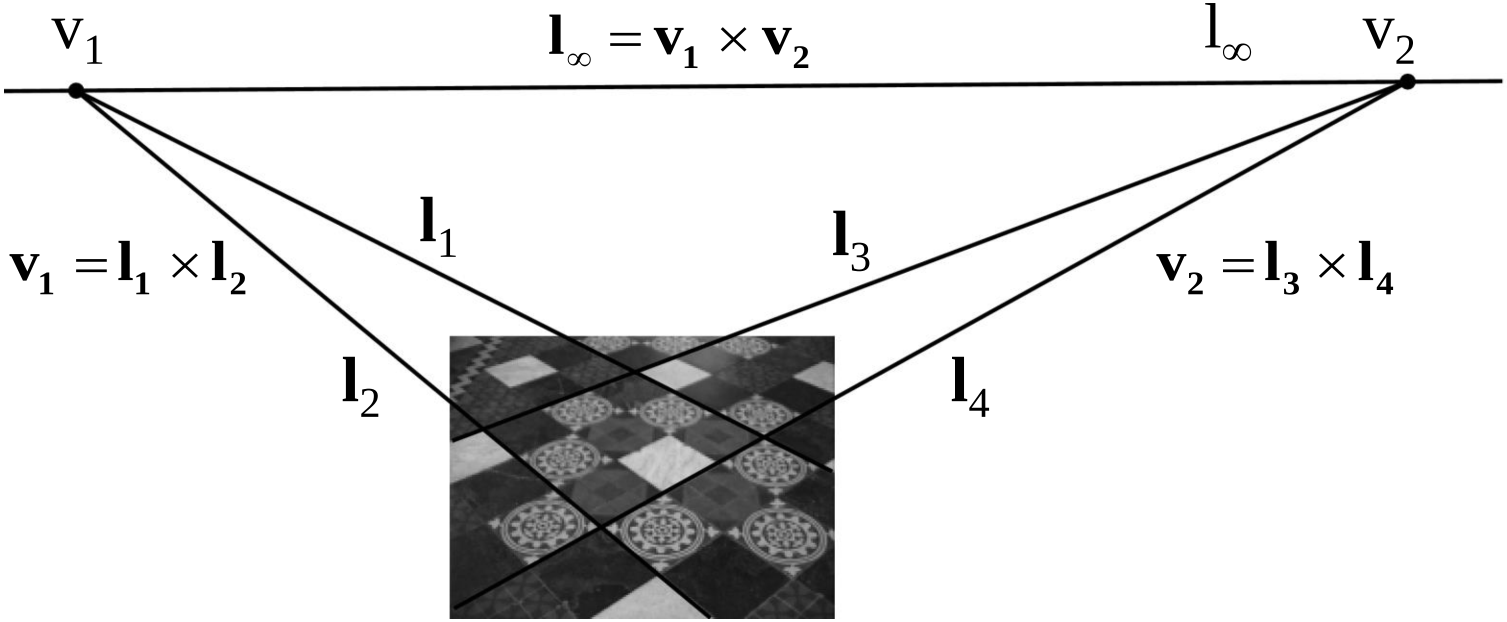
Affine Rectification



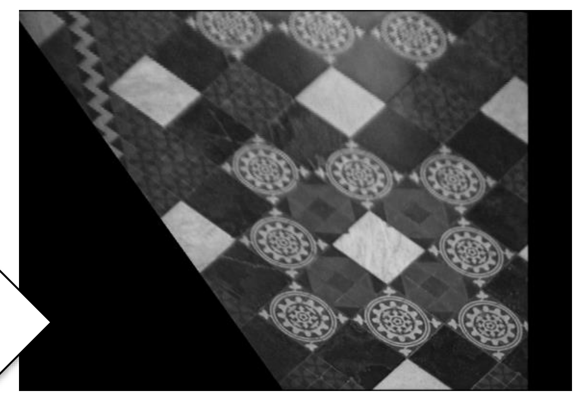
$$\begin{pmatrix}1 & & \\ & 1 & \\ l_1 & l_2 & l_3\end{pmatrix}^{-T}\begin{pmatrix}l_1\\ l_2\\ l_3\end{pmatrix} = \begin{pmatrix}1 & & -l_1/l_3\\ & 1 & -l_2/l_3\\ & & 1/l_3\end{pmatrix}\begin{pmatrix}l_1\\ l_2\\ l_3\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$

$$\begin{pmatrix}a_{11} & a_{12} & t_x\\ a_{21} & a_{22} & t_y\\ 0 & 0 & 1\end{pmatrix}$$

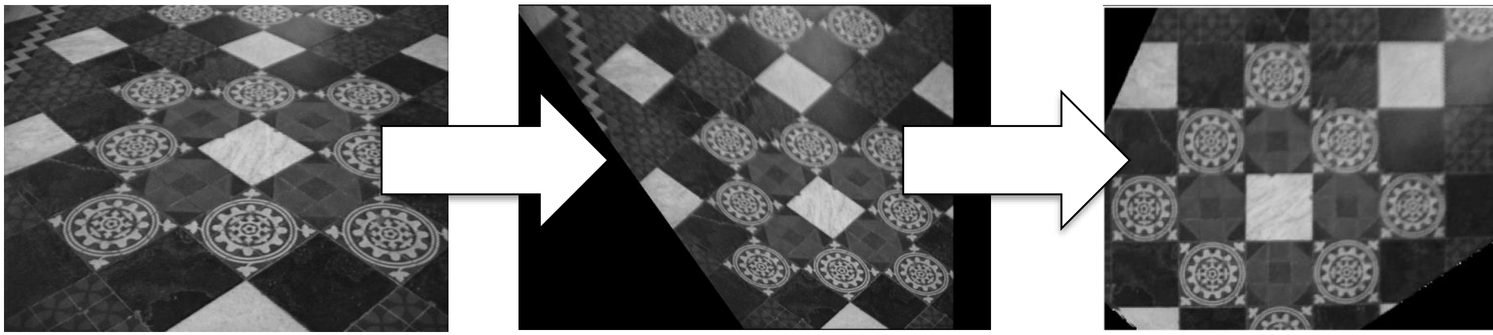
Affine Rectification



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}$$



Metric Rectification



- Need to measure a quantity that is not invariant under affine transformations

The Circular Points

$$\mathrm{I} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \qquad \mathrm{J} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

The circular points I, J are fixed points under the projective transformation **H** iff **H** is a similarity

$$\mathrm{I}' = \mathbf{H}_S\,\mathrm{I} = \begin{bmatrix} s\cos\theta & s\sin\theta & t_x \\ -s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = se^{i\theta} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \mathrm{I}$$

The Circular Points

- every circle intersects $\mathbf{l}_\infty$ at the “circular points”

$$x_1^2 + x_2^2 + d x_1 x_3 + e x_2 x_3 + f x_3^2 = 0$$

$$x_3 = 0$$

$$x_1^2 + x_2^2 = 0$$

$$\mathrm{I} = (1, i, 0)^\mathsf{T}$$

$$\mathrm{J} = (1, -i, 0)^\mathsf{T}$$

- Algebraically, encodes <u>orthogonal</u> directions

$$\mathrm{I} = (1,0,0)^\mathsf{T} + i(0,1,0)^\mathsf{T}$$

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Conic Dual to the Circular Points

$$\mathbf{C}^*_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{C}^*_\infty = \mathrm{I}\mathrm{J}^\mathsf{T} + \mathrm{J}\mathrm{I}^\mathsf{T}$$

$$\mathbf{C}^*_\infty = \mathbf{H}_S \mathbf{C}^*_\infty \mathbf{H}_S^\mathsf{T}$$

The dual conic $\mathbf{C}^*_\infty$ is fixed conic under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

Measuring Angles via the Dual Conic

- Euclidean: $\mathrm{l} = (l_1, l_2, l_3)^{\mathsf{T}} \qquad \mathrm{m} = (m_1, m_2, m_3)^{\mathsf{T}}$

$$\cos\theta = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}$$

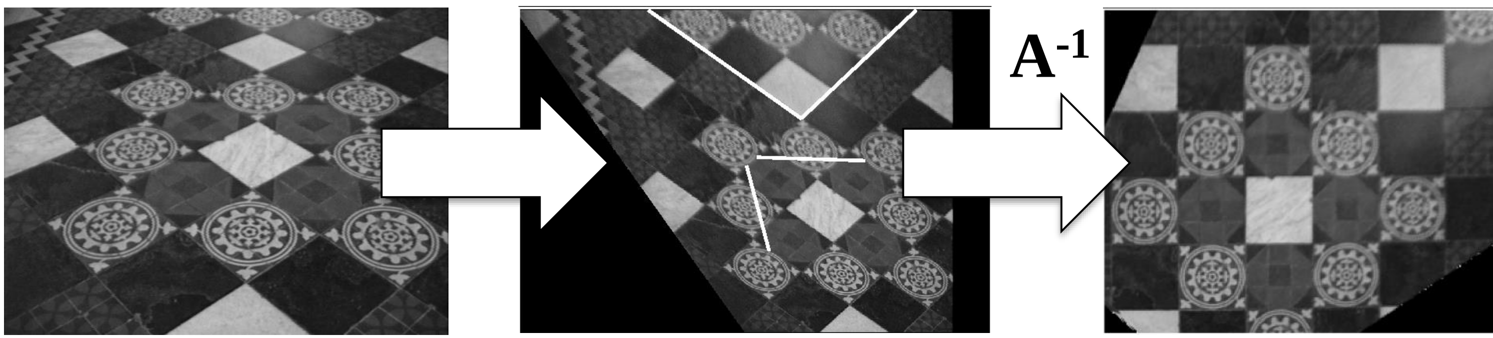
$$\mathbf{C}_\infty^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Projective: $\cos\theta = \dfrac{\mathrm{l}^{\mathsf{T}} \mathbf{C}_\infty^* \mathrm{m}}{\sqrt{(\mathrm{l}^{\mathsf{T}} \mathbf{C}_\infty^* \mathrm{l})(\mathrm{m}^{\mathsf{T}} \mathbf{C}_\infty^* \mathrm{m})}}$

$$\mathrm{l}^{\mathsf{T}} \mathbf{C}_\infty^* \mathrm{m} = 0 \quad \text{(orthogonal)}$$

- Knowing the dual conic on the projective plane, we can measure Euclidean angles!

Metric Rectification



- Dual conic under affinity

$$\mathbf{C}_\infty^{*}{}' = \begin{pmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{I} & 0 \\ 0^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{A}^T & 0 \\ \mathbf{t}^T & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{AA}^T & 0 \\ 0^T & 0 \end{pmatrix}$$

- $\mathbf{S}=\mathbf{AA}^\mathrm{T}$ symmetric, estimate from two pairs of orthogonal lines (due to $\mathrm{l}^\mathrm{T}\,\mathbf{C}_\infty^{*}\,\mathrm{m}=0$)

$$\left(l_1'm_1', l_1'm_2' + l_2'm_1', l_2'm_2'\right)\mathbf{s} = 0$$

Note: Result defined up to similarity

Update to Euclidean Space

- Metric space: Measure ratios of distances
- Euclidean space: Measure absolute distances
- Can we update metric to Euclidean space?
- Not without additional information

Important Points so far …

- Definition of 2D points and lines
- Definition of homogeneous coordinates
- Definition of projective space
- Effect of transformations on points, lines, conics
- Next: Analogous concepts in 3D

Overview

- 2D Projective Geometry
- **3D Projective Geometry**
- Camera Models & Calibration

ETH

3D Points and Planes

- 2D: duality point - line, 3D: duality point - plane

- Homogeneous representation of 3D points and planes

$$\pi_1 X_1 + \pi_2 X_2 + \pi_3 X_3 + \pi_4 X_4 = 0$$

- The point X lies on the plane π if and only if

$$\pi^{\mathsf{T}} \mathrm{X} = 0$$

- The plane π goes through the point X if and only if

$$\pi^{\mathsf{T}} \mathrm{X} = 0$$

Planes from Points

Solve π from $X_1^{\mathsf{T}}\pi = 0$, $X_2^{\mathsf{T}}\pi = 0$ and $X_3^{\mathsf{T}}\pi = 0$

$$\begin{bmatrix} X_1^{\mathsf{T}} \\ X_2^{\mathsf{T}} \\ X_3^{\mathsf{T}} \end{bmatrix} \pi = 0$$

(solve π as right nullspace of $\begin{bmatrix} X_1^{\mathsf{T}} \\ X_2^{\mathsf{T}} \\ X_3^{\mathsf{T}} \end{bmatrix}$)

Points from Planes

Solve X from $\pi_1^{\mathsf{T}}\mathrm{X} = 0$, $\pi_2^{\mathsf{T}}\mathrm{X} = 0$ and $\pi_3^{\mathsf{T}}\mathrm{X} = 0$

$$\begin{bmatrix} \pi_1^{\mathsf{T}} \\ \pi_2^{\mathsf{T}} \\ \pi_3^{\mathsf{T}} \end{bmatrix} \mathrm{X} = 0$$

(solve X as right nullspace of $\begin{bmatrix} \pi_1^{\mathsf{T}} \\ \pi_2^{\mathsf{T}} \\ \pi_3^{\mathsf{T}} \end{bmatrix}$)

Representing a plane by its span

$$\mathrm{X} = \mathbf{M}\,\mathrm{x} \qquad \mathbf{M} = \left[\mathbf{X}_1\ \mathbf{X}_2\ \mathbf{X}_3\right] \in R^{4\times 3}$$

$$\pi^{\mathsf{T}}\mathbf{M} = 0$$

Quadrics and Dual Quadrics

$\mathbf{X}^{\mathsf{T}}\mathbf{Q}\mathbf{X}=\mathbf{0}$ ($\mathbf{Q}$: 4x4 symmetric matrix)

- 9 DOF (up to scale)
- In general, 9 points define quadric
- det($\mathbf{Q}$)=0 ↔ degenerate quadric
- tangent plane $p=\mathbf{Q}\mathbf{X}$
- Dual quadric: $p^{\mathsf{T}}\mathbf{Q}^{*}p=\mathbf{0}$ ($\mathbf{Q}^{*}$ adjoint)
- relation to quadric $\mathbf{Q}^{*}=\mathbf{Q}^{-1}$ (non-degenerate)

$$\mathbf{Q}=\begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \\ \circ & \circ & \circ & \bullet \end{bmatrix}$$

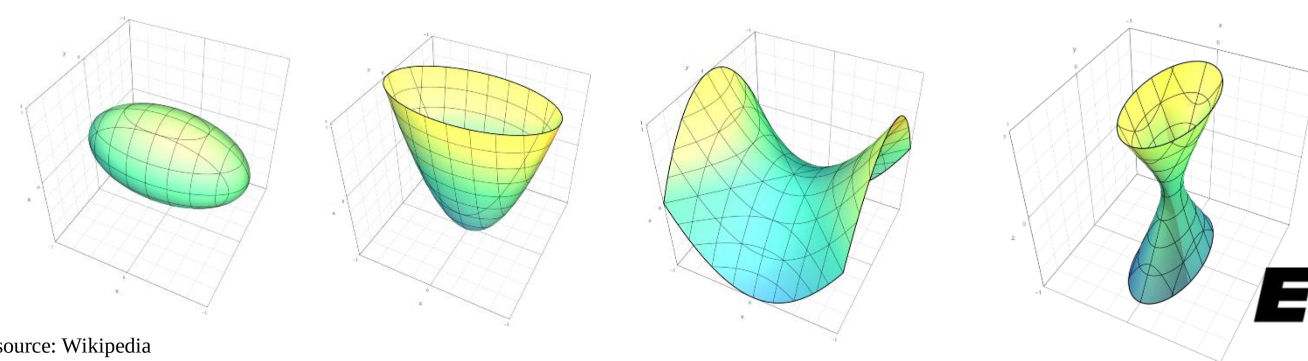


Image source: Wikipedia

Transformation of 3D points, planes and quadrics

- Transformation for points (2D equivalent)

$$\mathrm{X}' = \mathbf{H}\,\mathrm{X} \qquad (\mathrm{x}' = \mathbf{H}\,\mathrm{x})$$

- Transformation for planes

$$\pi' = \mathbf{H}^{-\mathsf{T}}\pi \qquad (\mathrm{l}' = \mathbf{H}^{-\mathsf{T}}\,\mathrm{l})$$

- Transformation for quadrics

$$\mathbf{Q}' = \mathbf{H}^{-\mathsf{T}}\mathbf{Q}\mathbf{H}^{-1} \qquad (\mathbf{C}' = \mathbf{H}^{-\mathsf{T}}\mathbf{C}\mathbf{H}^{-1})$$

- Transformation for dual quadrics

$$\mathbf{Q}'^{*} = \mathbf{H}\mathbf{Q}^{*}\mathbf{H}^{\mathsf{T}} \qquad (\mathbf{C}'^{*} = \mathbf{H}\mathbf{C}^{*}\mathbf{H}^{\mathsf{T}})$$

The Plane at Infinity

The plane at infinity $\pi_\infty=(0, 0, 0, 1)^\mathsf{T}$ is a fixed plane under a projective transformation H iff H is an affinity

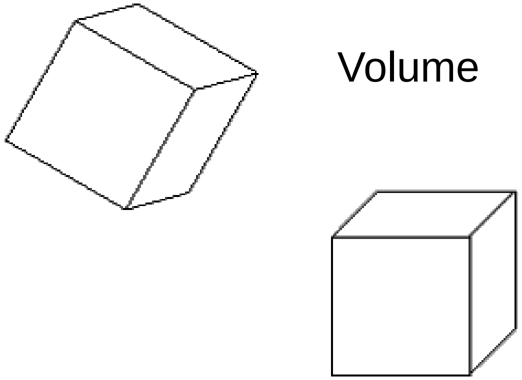
$$\pi'_\infty = \mathbf{H}_A^{-\mathsf{T}}\pi_\infty = \begin{bmatrix} \mathbf{A}^{-\mathsf{T}} & 0 \\ \text{-}\mathrm{t}^\mathsf{T}\mathbf{A}^{-\mathsf{T}} & 1 \end{bmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \pi_\infty$$

1. canonical position $p_{¥} = (0,0,0,1)^\mathsf{T}$
2. contains all directions $\mathrm{D} = (X_1, X_2, X_3, 0)^\mathsf{T}$
3. two planes are parallel ⇔ line of intersection in π_∞
4. line || line (or plane) ⇔ point of intersection in π_∞
5. 2D equivalent: line at infinity

Hierarchy of 3D Transformations

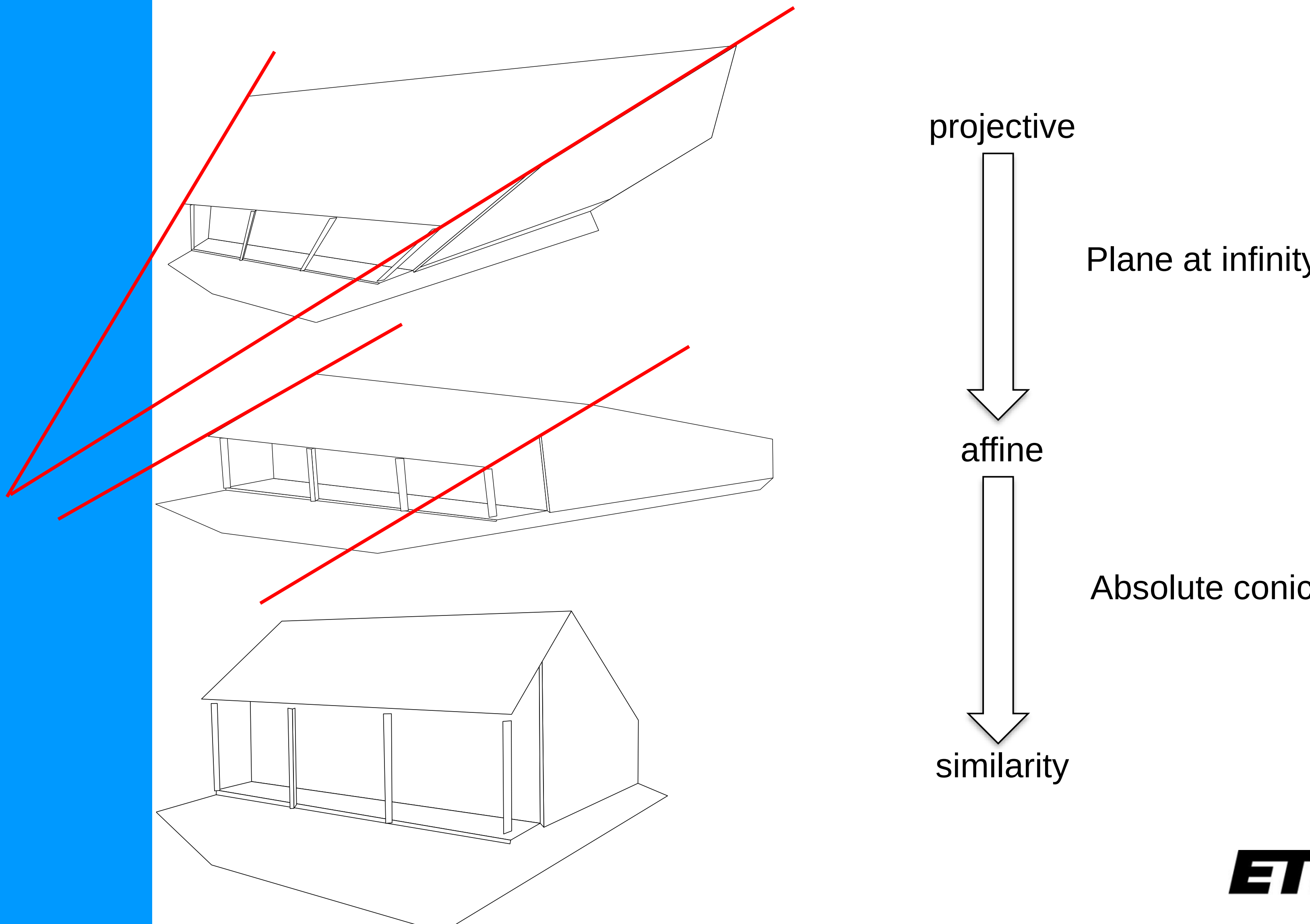
Euclidean
6dof

$$\begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^{T} & 1 \end{bmatrix}$$



Volume

Hierarchy of 3D Transformations



The Absolute Conic

- The absolute conic Ω_∞ is a (point) conic on π_∞
- In a metric frame: $$\left.\begin{array}{c} X_1^2 + X_2^2 + X_3^2 \\ X_4 \end{array}\right\} = 0$$

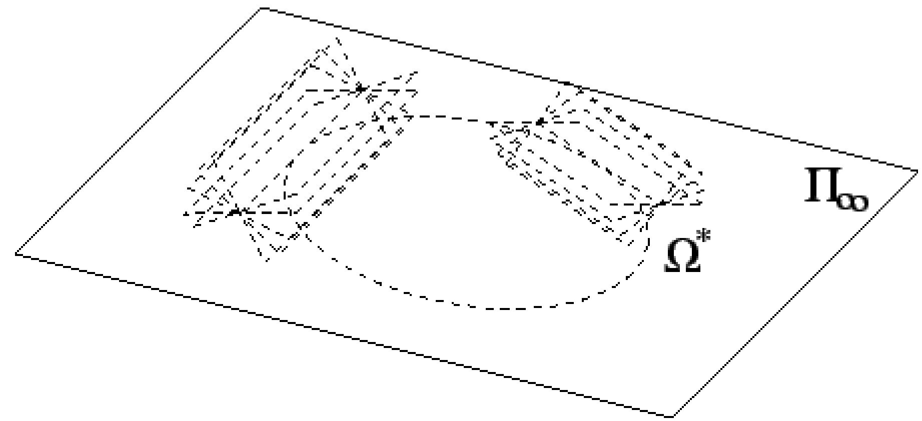
  or conic for directions: $$(X_1, X_2, X_3)\mathbf{I}(X_1, X_2, X_3)^\mathsf{T}$$
  (with no real points)

> The absolute conic Ω_∞ is a fixed conic under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

1. Ω_∞ is only fixed as a set
2. Circles intersect Ω_∞ in two circular points
3. Spheres intersect π_∞ in Ω_∞

The Absolute Dual Quadric

$$\Omega^*_\infty = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix}$$



The absolute dual quadric Ω^*_∞ is a fixed quadric under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

1. 8 dof
2. plane at infinity π_∞ is the nullvector of Ω_∞
3. angles: $$\cos\theta = \frac{\pi_1^\mathsf{T}\Omega^*_\infty\pi_2}{\sqrt{\left(\pi_1^\mathsf{T}\Omega^*_\infty\pi_1\right)\left(\pi_2^\mathsf{T}\Omega^*_\infty\pi_2\right)}}$$

Important Points so far …

- Def. of 2D points and lines, 3D points and planes
- Def. of homogeneous coordinates
- Def. of projective space (2D and 3D)
- Effect of transformations on points, lines, planes
- Next: Projections from 3D to 2D

ETH

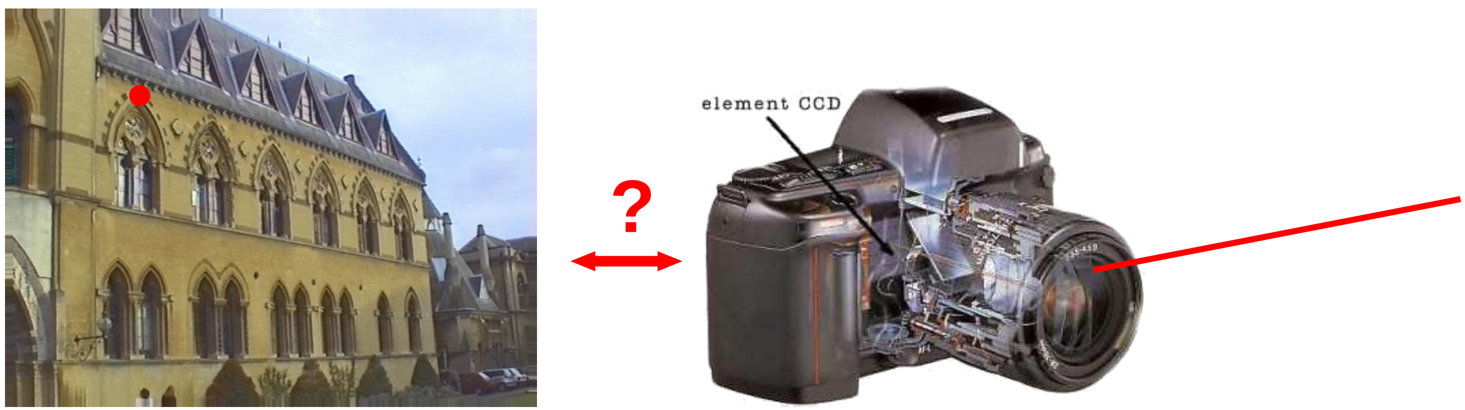
Overview

- 2D Projective Geometry
- 3D Projective Geometry
- **Camera Models & Calibration**

ETH

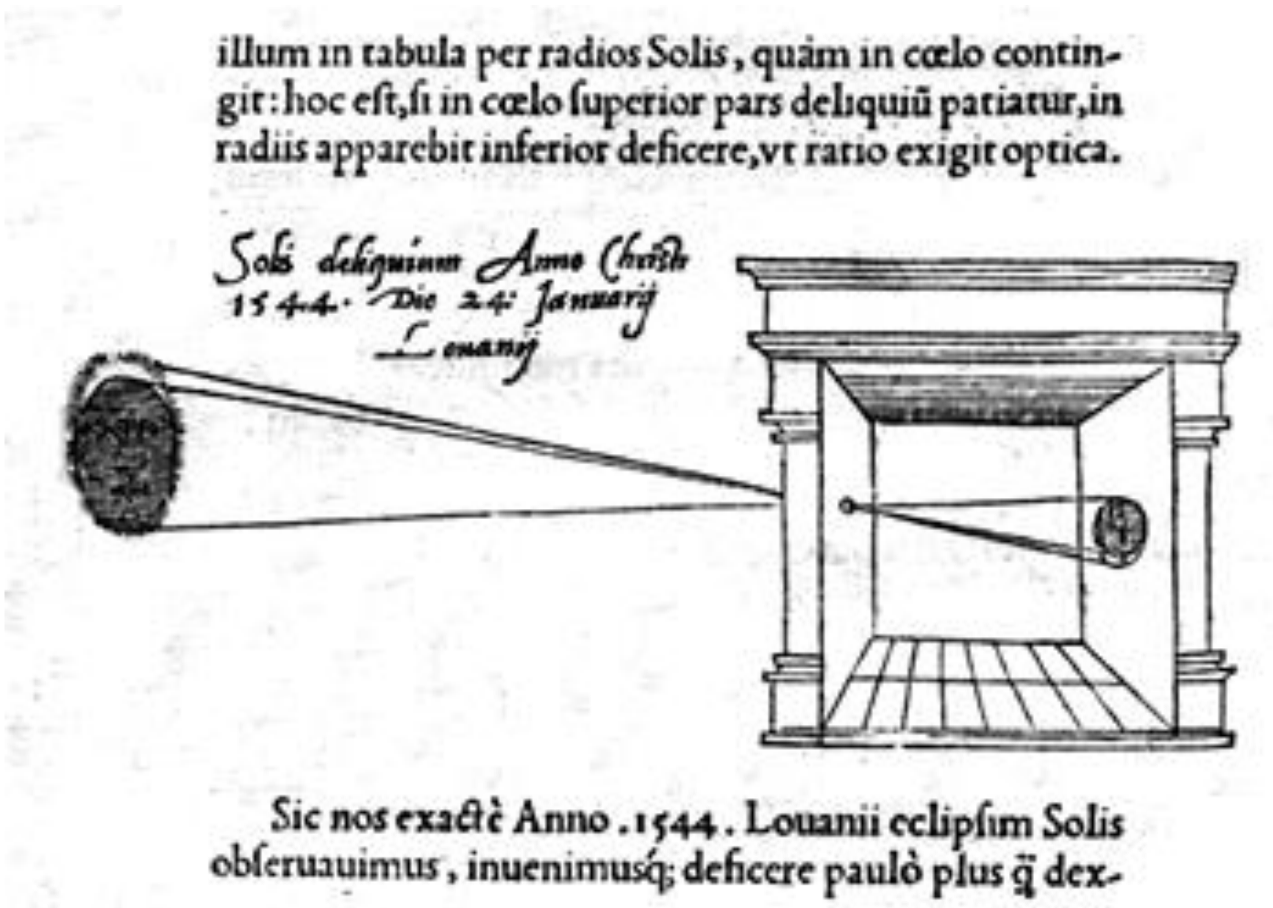
Camera Model

Relation between pixels and rays in space



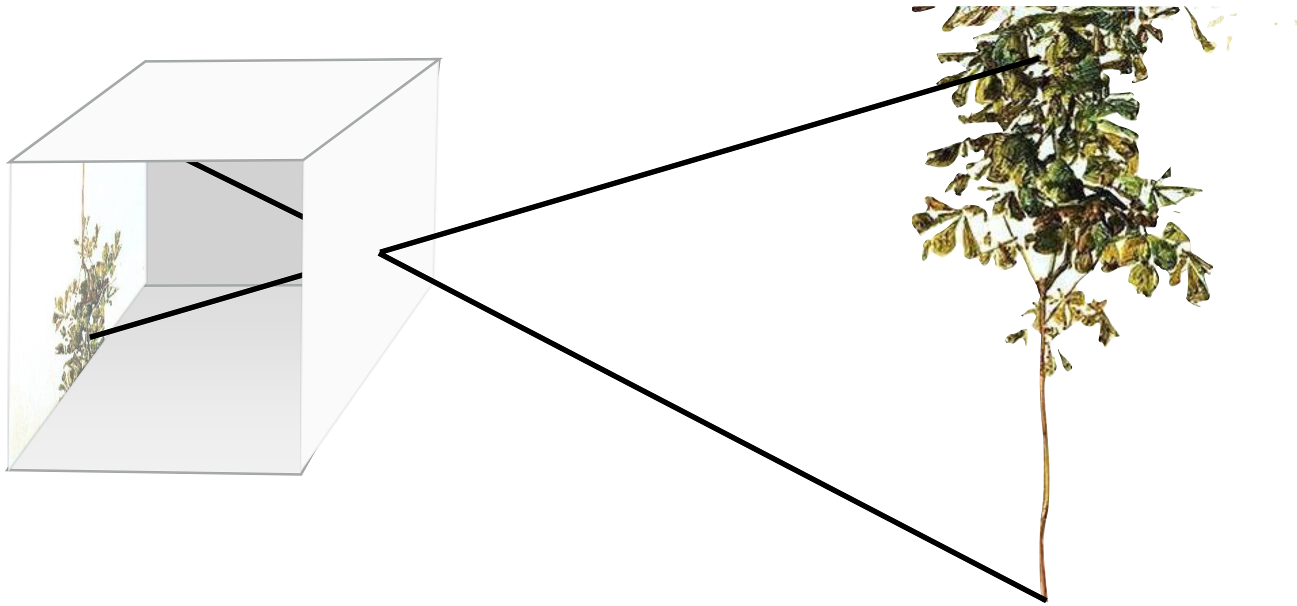
ETH

Pinhole Camera


illum in tabula per radios Solis, quàm in cœlo contingit: hoc eſt, ſi in cœlo ſuperior pars deliquiũ patiatur, in radiis apparebit inferior deficere, vt ratio exigit optica.

Sic nos exactè Anno .1544. Louanii eclipſim Solis obſeruauimus, inuenimusq́; deficere paulò plus q̃ dex-

Pinhole Camera



Pinhole Camera

Pinhole Camera

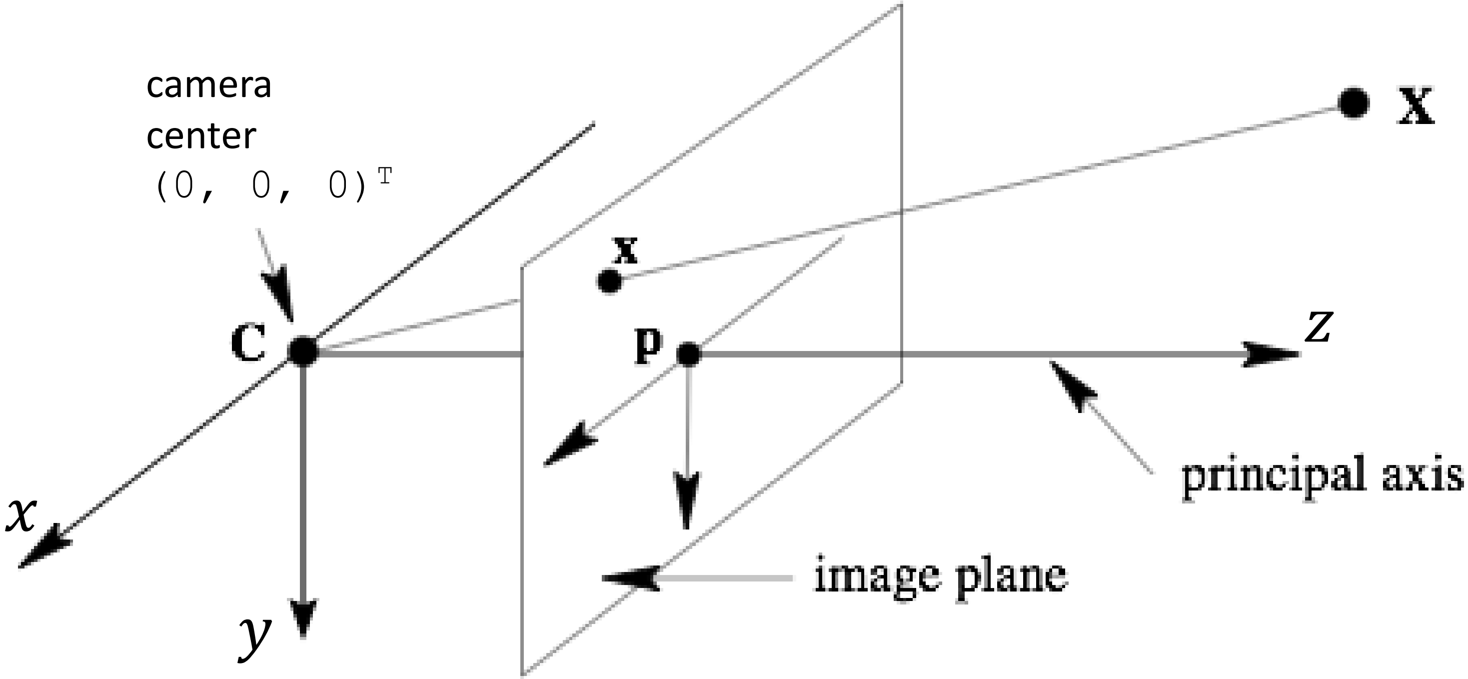
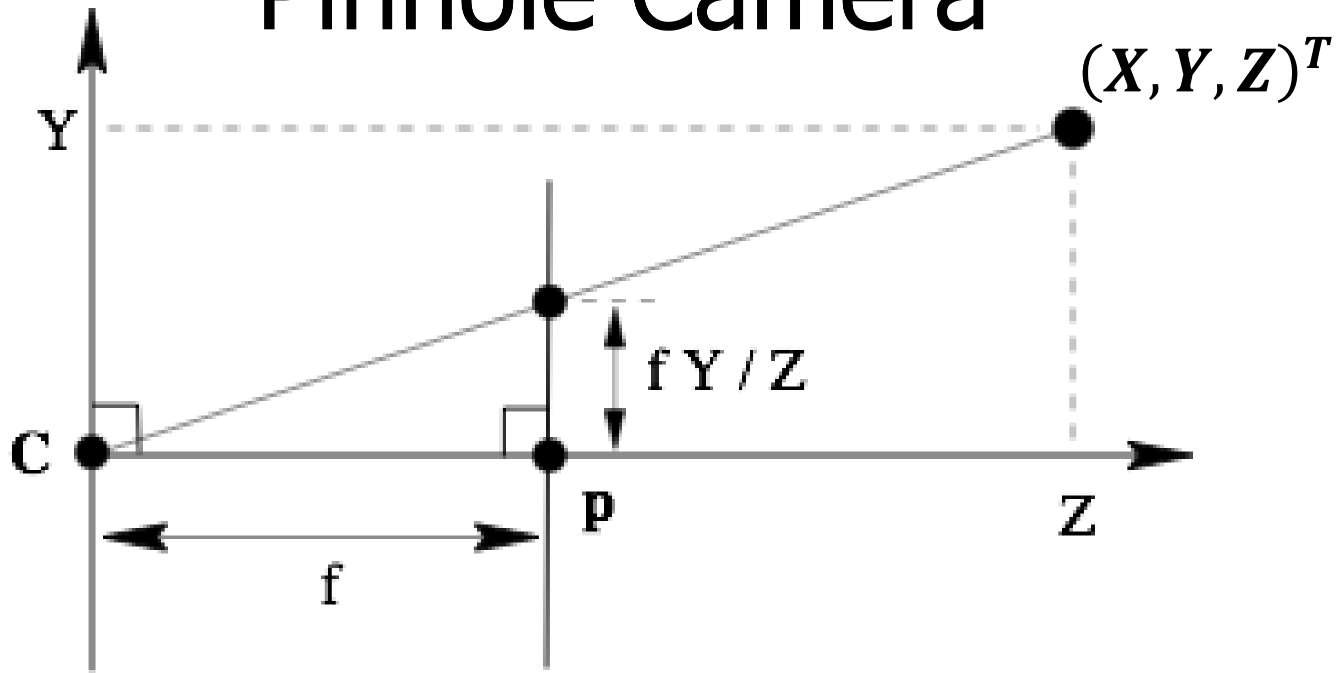


figure adapted from Hartley and Zisserman, 2004

Pinhole Camera



Projection as matrix multiplication:

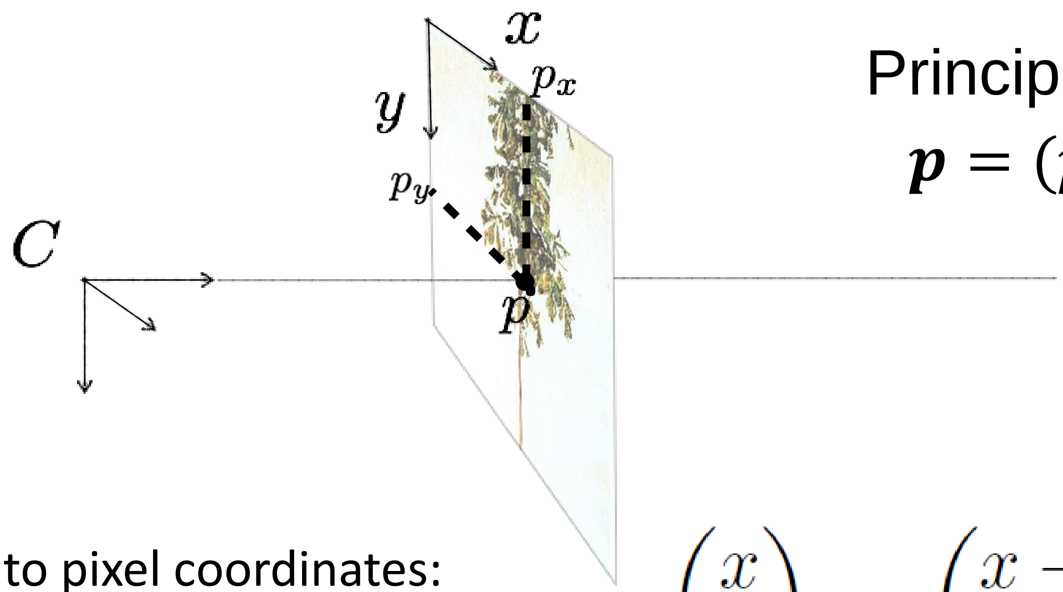
$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{pmatrix} fX/Z \\ fY/Z \\ 1 \end{pmatrix}$$

De-homogenization:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x'/z' \\ y'/z' \end{pmatrix}$$

ETH

Pinhole Camera



Principal point
$\boldsymbol{p} = (p_x, p_y)$

Mapping to pixel coordinates:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + p_x \\ y + p_y \end{pmatrix}$$

Projection as matrix multiplication:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

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Intrinsic Camera Parameters

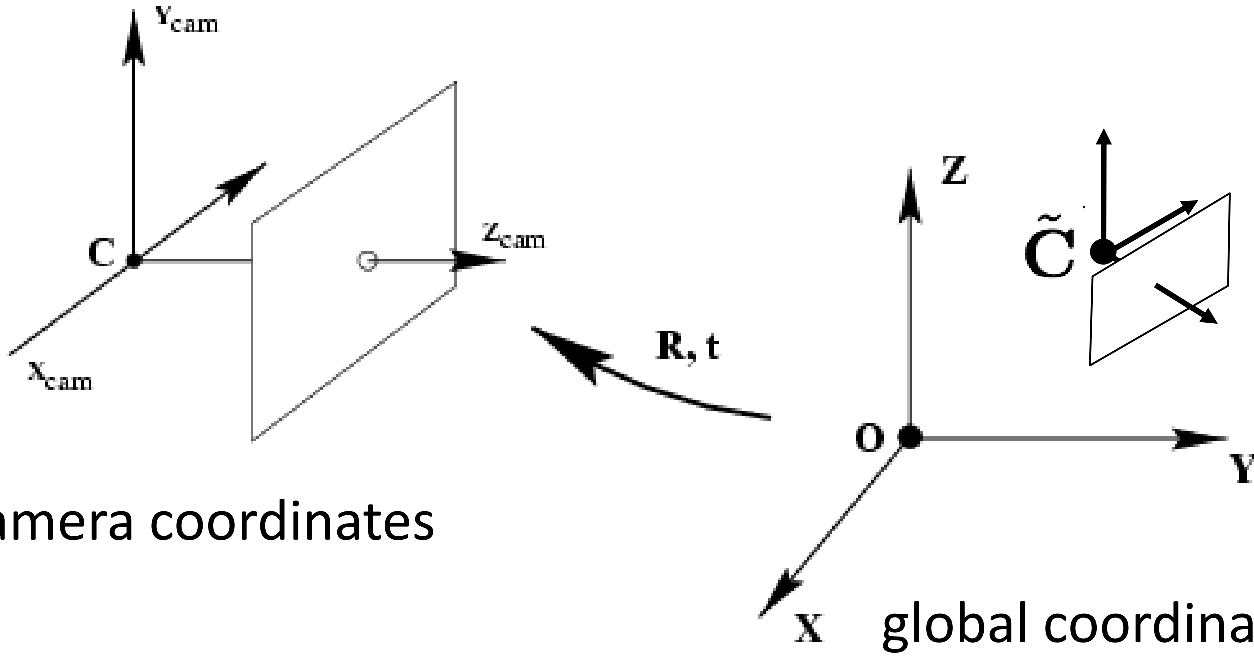
General intrinsic camera calibration matrix:

$$\mathrm{K} = \begin{pmatrix} f & s & p_x \\ 0 & \alpha f & p_y \\ 0 & 0 & 1 \end{pmatrix}$$

In practice:

$$\mathrm{K} = \begin{pmatrix} f & 0 & w/2 \\ 0 & f & h/2 \\ 0 & 0 & 1 \end{pmatrix}$$

Extrinsic Camera Parameters



camera coordinates

global coordinates

Transformation from global to camera coordinates:

$$\mathbf{X}_{\text{cam}} = \mathrm{R}\left(\mathbf{X}_{\text{global}} - \tilde{\mathbf{C}}\right)$$

figure adapted from Hartley and Zisserman, 2004

Projection Matrix

Projection from 3D global coordinates to pixels:

$$\lambda \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \mathtt{K}\left(\mathtt{R}\mathbf{X}_{\text{global}} + \mathbf{t}\right)$$

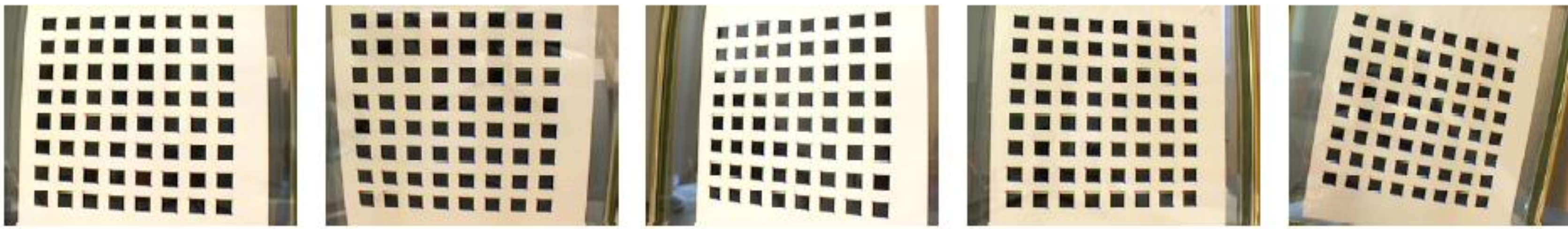
3x4 matrix
(maps from $\mathbb{P}^3$ to $\mathbb{P}^2$)

projection matrix

figure adapted from Hartley and Zisserman, 2004

Practical Camera Calibration

Method and Pictures from Zhang (ICCV' 99): "Flexible Camera Calibration By Viewing a Plane From Unknown Orientations"



Unknown: constant camera intrinsics **K**
(varying) camera poses **R,t**

Known: 3D coordinates of chessboard corners
=> Define to be the **z=0** plane ($\mathbf{X}=[X_1\ X_2\ 0\ 1]^T$)

Point is mapped as $\lambda \mathbf{x} = \mathbf{K}\,(\mathbf{r}_1\ \mathbf{r}_2\ \mathbf{r}_3\ \mathbf{t})\,\mathbf{X}$

$\lambda \mathbf{x} = \mathbf{K}\,(\mathbf{r}_1\ \mathbf{r}_2\ \mathbf{t})\,[X_1\ X_2\ 1]'$

$$\mathbf{K} = \begin{bmatrix} f_x & & p_x \\ & f_y & p_y \\ & & 1 \end{bmatrix}$$

Homography **H** between image and chess coordinates, estimate from known $\mathbf{X}_i$ and measured $\mathbf{x}_i$

ETH

Direct Linear Transformation (DLT)

$$\mathbf{x}'_i \times \mathbf{H}\mathbf{x}_i = 0 \qquad \mathbf{x}'_i = \left(x'_i, y'_i, w'_i\right)^\top \qquad \mathbf{H}\mathbf{x}_i = \begin{pmatrix} \mathbf{h}^{1\top}\mathbf{x}_i \\ \mathbf{h}^{2\top}\mathbf{x}_i \\ \mathbf{h}^{3\top}\mathbf{x}_i \end{pmatrix}$$

$$\mathbf{x}'_i \times \mathbf{H}\mathbf{x}_i = \begin{pmatrix} y'_i\mathbf{h}^{3\top}\mathbf{x}_i - w'_i\mathbf{h}^{2\top}\mathbf{x}_i \\ w'_i\mathbf{h}^{1\top}\mathbf{x}_i - x'_i\mathbf{h}^{3\top}\mathbf{x}_i \\ x'_i\mathbf{h}^{2\top}\mathbf{x}_i - y'_i\mathbf{h}^{1\top}\mathbf{x}_i \end{pmatrix}$$

$$\begin{bmatrix} \mathbf{0}^\top & -w'_i\mathbf{x}_i^\top & y'_i\mathbf{x}_i^\top \\ w'_i\mathbf{x}_i^\top & \mathbf{0}^\top & -x'_i\mathbf{x}_i^\top \\ -y'_i\mathbf{x}_i^\top & x'_i\mathbf{x}_i^\top & \mathbf{0}^\top \end{bmatrix} \begin{pmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{pmatrix} = \mathbf{0}$$

Direct Linear Transformation (DLT)

- Equations are linear in $\mathbf{h}$: $\mathbf{A}_i\mathbf{h} = 0$
- Only 2 out of 3 are linearly independent (2 equations per point)

$$\begin{bmatrix} \mathbf{0}^\top & -w'_i\mathbf{x}_i^\top & y'_i\mathbf{x}_i^\top \\ w'_i\mathbf{x}_i^\top & \mathbf{0}^\top & -x'_i\mathbf{x}_i^\top \\ -y'_i\mathbf{x}_i^\top & x'_i\mathbf{x}_i^\top & \mathbf{0}^\top \end{bmatrix}\begin{pmatrix}\mathbf{h}^1\\ \mathbf{h}^2\\ \mathbf{h}^3\end{pmatrix} = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{0}^\top & -w'_i\mathbf{x}_i^\top & y'_i\mathbf{x}_i^\top \\ w'_i\mathbf{x}_i^\top & \mathbf{0}^\top & -x'_i\mathbf{x}_i^\top \end{bmatrix}\begin{pmatrix}\mathbf{h}^1\\ \mathbf{h}^2\\ \mathbf{h}^3\end{pmatrix} = \mathbf{0}$$

(only drop third row if $w_i' \neq 0$)

- Holds for any homogeneous representation, e.g. $(x_i', y_i', 1)$

Direct Linear Transformation (DLT)

- Solving for homography $\mathbf{H}$

$$\begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \mathbf{A}_3 \\ \mathbf{A}_4 \end{bmatrix} \mathbf{h} = \mathbf{0} \qquad \mathbf{A}\mathbf{h} = \mathbf{0}$$

size A is 8x9 (2eq.) or 12x9 (3eq.), but rank 8

- Trivial solution is $\mathrm{h}=0_9^T$ is not interesting
- 1D null-space yields solution of interest

pick for example the one with $\|\mathbf{h}\| = 1$

ETH

Direct Linear Transformation (DLT)

- Over-determined solution

$$\begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_n \end{bmatrix} \mathbf{h} = \mathbf{0}$$

  - No exact solution because of inexact measurement, i.e., “noise”
  - Find approximate solution
    - Additional constraint needed to avoid $\mathbf{0}$, e.g., $\|\mathbf{h}\| = 1$
    - $\mathbf{Ah} = \mathbf{0}$ not possible, so minimize $\|\mathbf{Ah}\|$

DLT Algorithm

Objective

Given $n \geq 4$ 2D to 2D point correspondences $\{\mathbf{x}_i \leftrightarrow \mathbf{x}_i'\}$, determine the 2D homography matrix **H** such that $\mathbf{x}_i' = \mathbf{H}\mathbf{x}_i$

Algorithm

(i) For each correspondence $\mathbf{x}_i \leftrightarrow \mathbf{x}_i'$ compute $\mathbf{A}_i$. Usually only two first rows needed.

(ii) Assemble n 2x9 matrices $\mathbf{A}_i$ into a single $2n$x9 matrix **A**

(iii) Obtain SVD of **A**. Solution for **h** is last column of V

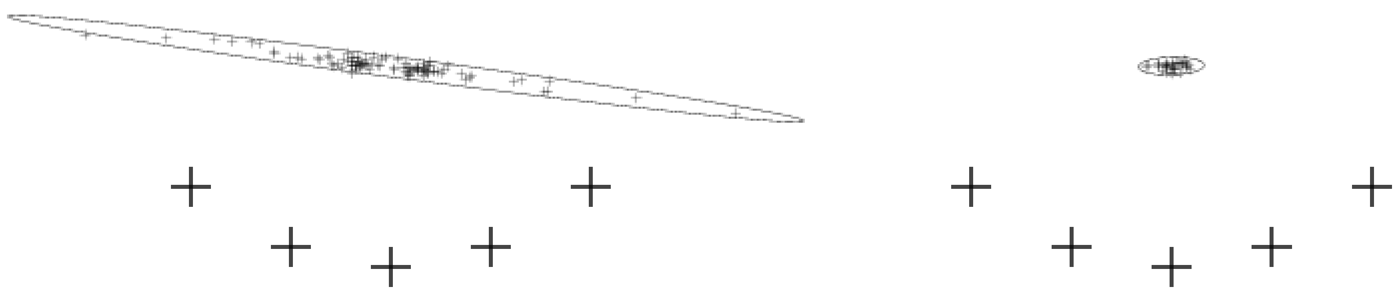
(iv) Determine **H** from **h**

Importance of Normalization

$$\begin{bmatrix} 0 & 0 & 0 & -x_i & -y_i & -1 & y_i'x_i & y_i'y_i & y_i' \\ x_i & y_i & 1 & 0 & 0 & 0 & -x_i'x_i & -x_i'y_i & -x_i' \end{bmatrix} \begin{pmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{pmatrix} = \mathbf{0}$$

$\sim 10^2$ $\sim 10^2$ 1 $\sim 10^2$ $\sim 10^2$ 1 $\sim 10^4$ $\sim 10^4$ $\sim 10^2$

orders of magnitude difference!



Monte Carlo simulation

for identity computation based on 5 points

(not normalized ↔ normalized)

Normalized DLT Algorithm

Objective

Given n≥4 2D to 2D point correspondences $\{\mathbf{x}_i \leftrightarrow \mathbf{x}_i'\}$, determine the 2D homography matrix **H** such that $\mathbf{x}_i' = \mathbf{H}\mathbf{x}_i$

Algorithm

(i) Normalize points $\tilde{\mathbf{x}}_i = \mathbf{T}_{\mathbf{norm}}\mathbf{x}_i, \tilde{\mathbf{x}}_i' = \mathbf{T}_{\mathbf{norm}}'\mathbf{x}_i'$

(ii) Apply DLT algorithm to $\tilde{\mathbf{x}}_i \leftrightarrow \tilde{\mathbf{x}}_i'$,

(iii) Denormalize solution $\mathbf{H} = \mathbf{T}_{\mathbf{norm}}'^{-1}\tilde{\mathbf{H}}\mathbf{T}_{\mathbf{norm}}$

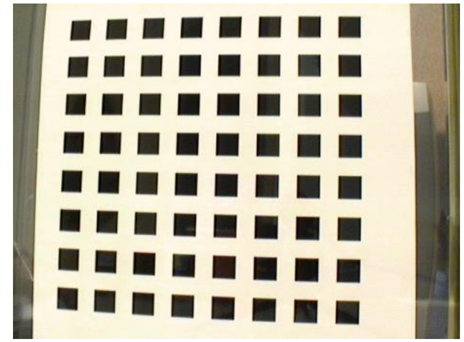
Normalization (independently per image):

- Translate points such that centroid is at origin
- Isotropic scaling such that mean distance to origin is $\sqrt{2}$

ETH

Geometric Distance

- x measured coordinates
- $\hat{\mathrm{x}}$ estimated coordinates
- $\bar{\mathrm{x}}$ true coordinates
- $d(.,.)$ Euclidean distance (in image)



e.g. calibration pattern

Error in one image

$$\hat{\mathbf{H}} = \underset{\mathbf{H}}{\mathbf{argmin}} \sum_i d\left(\mathbf{x}'_i, \mathbf{H}\bar{\mathbf{x}}_i\right)^2$$

Symmetric transfer error

$$\hat{\mathbf{H}} = \underset{\mathbf{H}}{\mathbf{argmin}} \sum_i d\left(\mathbf{x}_i, \mathbf{H}^{-1}\mathbf{x}'_i\right)^2 + d\left(\mathbf{x}'_i, \mathbf{H}\mathbf{x}_i\right)^2$$

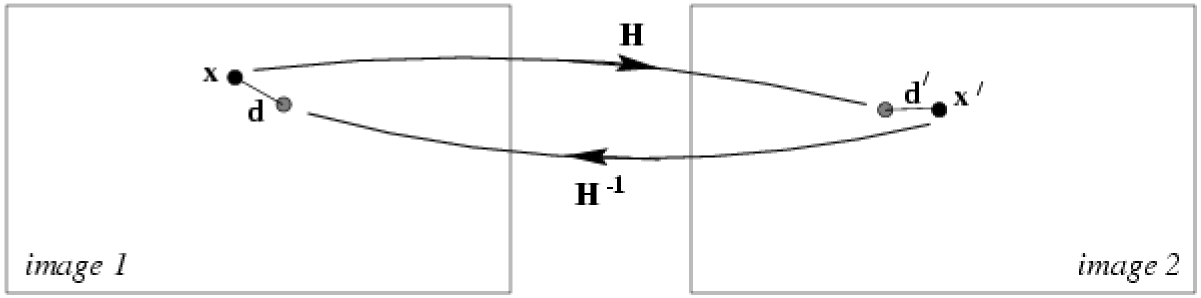
Reprojection error

$$\left(\hat{\mathbf{H}}, \hat{\mathbf{x}}_i, \hat{\mathbf{x}}'_i\right) = \underset{\hat{\mathbf{H}}, \hat{\mathbf{x}}_i, \hat{\mathbf{x}}'_i}{\mathbf{argmin}} \sum_i d\left(\mathbf{x}_i, \hat{\mathbf{x}}_i\right)^2 + d\left(\mathbf{x}'_i, \hat{\mathbf{x}}'_i\right)^2$$

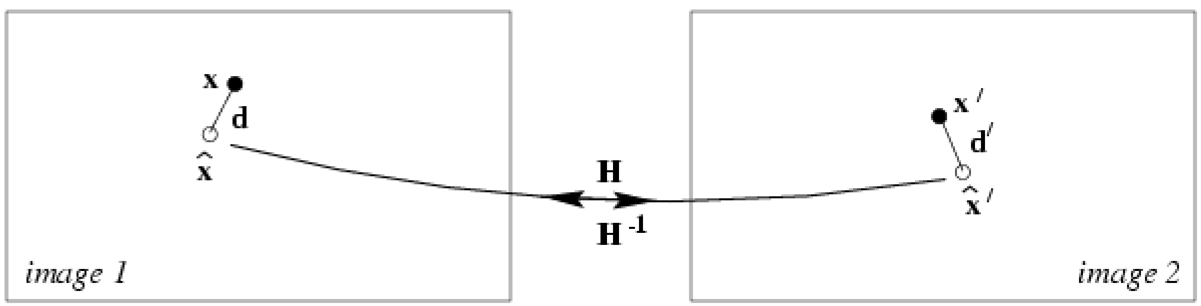
subject to $\hat{\mathbf{x}}_i = \hat{\mathbf{H}}\hat{\mathbf{x}}'_i$

ETH

Reprojection Error



$$d\left(\mathbf{x},\mathbf{H}^{-1}\mathbf{x}'\right)^2 + d\left(\mathbf{x}',\mathbf{H}\mathbf{x}\right)^2$$



$$d\left(\mathbf{x},\hat{\mathbf{x}}\right)^2 + d\left(\mathbf{x}',\hat{\mathbf{x}}'\right)^2$$

Statistical Cost Function and Maximum Likelihood Estimation

- Optimal cost function related to noise model
- Assume zero-mean isotropic Gaussian noise (assume outliers removed)

$$\Pr(\mathrm{x}) = \frac{1}{2\pi\sigma^2} e^{-d(\mathrm{x},\bar{\mathrm{x}})^2/(2\sigma^2)}$$

Error in one image

$$\Pr(\{\mathbf{x}'_\mathbf{i}\} \mid \mathbf{H}) = \prod_i \frac{1}{2\pi\sigma^2} e^{-d(\mathbf{x}'_\mathbf{i},\mathbf{H}\bar{\mathbf{x}}_i)^2/(2\sigma^2)}$$

$$\log \Pr(\{\mathbf{x}'_\mathbf{i}\} \mid \mathbf{H}) = -\frac{1}{2\sigma^2} \sum_i d(\mathbf{x}'_\mathbf{i},\mathbf{H}\bar{\mathbf{x}}_\mathbf{i})^2 + const$$

Maximum Likelihood Estimate:

$$\min \sum d(\mathbf{x}'_\mathbf{i},\mathbf{H}\bar{\mathbf{x}}_i)^2$$

ETH

Gold Standard Algorithm

Objective

Given n≥4 2D to 2D point correspondences $\{\mathbf{x}_i \leftrightarrow \mathbf{x}_i'\}$, determine the Maximum Likelihood Estimation of **H**

(this also implies computing optimal $\mathbf{x}_i' = \mathbf{H}\mathbf{x}_i$)

Algorithm

(i) **Initialization:** compute an initial estimate using normalized DLT or RANSAC

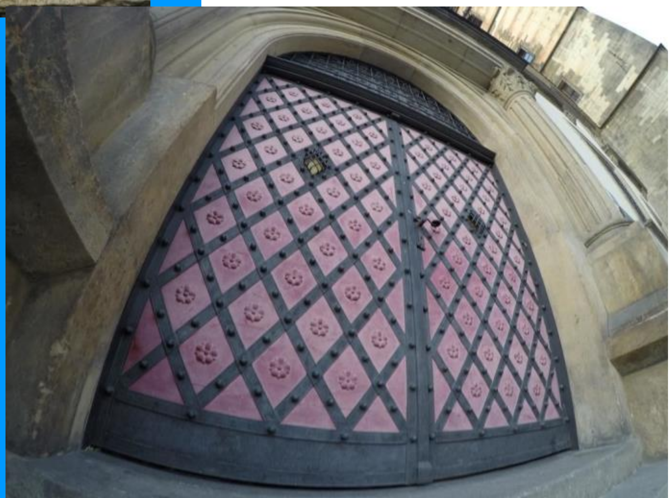
(ii) **Geometric minimization of symmetric transfer error:**

- Minimize using Levenberg-Marquardt over 9 entries of **h**

or reprojection error:

- compute initial estimate for optimal $\{\mathbf{x}_i\}$
- minimize cost $\sum d(\mathbf{x}_i, \hat{\mathbf{x}}_i)^2 + d(\mathbf{x}_i', \hat{\mathbf{x}}_i')^2$ over $\{\mathbf{H}, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$
- if many points, use sparse method

Radial Distortion



straight lines are not straight anymore

- Due to spherical lenses (cheap)
- (One possible) model:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f_x & s & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix} \mathrm{R} \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{R}^\top & -\mathbf{R}^\top \mathbf{t} \\ 0_3^\top & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \right)$$

$$\mathrm{R}: \quad (x, y) = (1 + K_1(x^2 + y^2) + K_2(x^2 + y^2)^2 + \ldots) \begin{bmatrix} x \\ y \end{bmatrix}$$

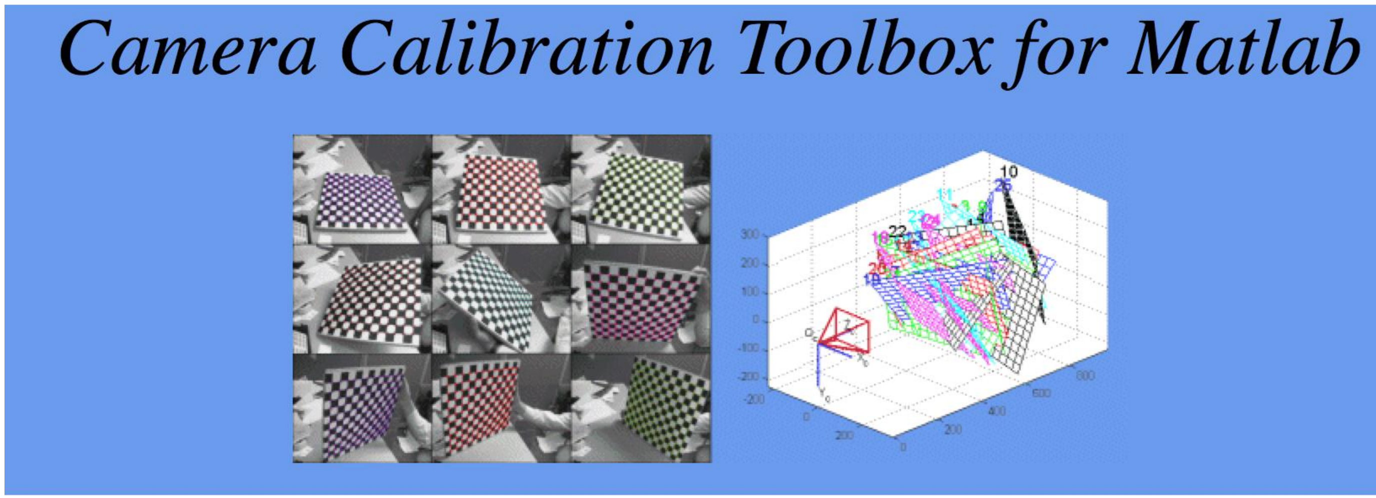
ETH

Calibration with Radial Distortion

- Low radial distortion:
  - Ignore radial distortion during initial calibration
  - Estimate distortion parameters, refine full calibration

- High radial distortion: Simultaneous estimation
  - Fitzgibbon, “Simultaneous linear estimation of multiple view geometry and lens distortion”, CVPR 2001
  - Kukelova et al., “Real-Time Solution to the Absolute Pose Problem with Unknown Radial Distortion and Focal Length”, ICCV 2013
  - Larsson et al., “Revisiting Radial Distortion Absolute Pose”, ICCV 2019

Bouguet Toolbox



http://www.vision.caltech.edu/bouguetj/calib_doc/

Rolling Shutter Cameras

- Image build row by row
- Distortions based on depth and speed
- Many mobile phone cameras have rolling shutter

Video credit: Olivier Saurer

Rolling Shutter Effect

Global shutter Rolling shutter

Slide credit:
Cenek Albl

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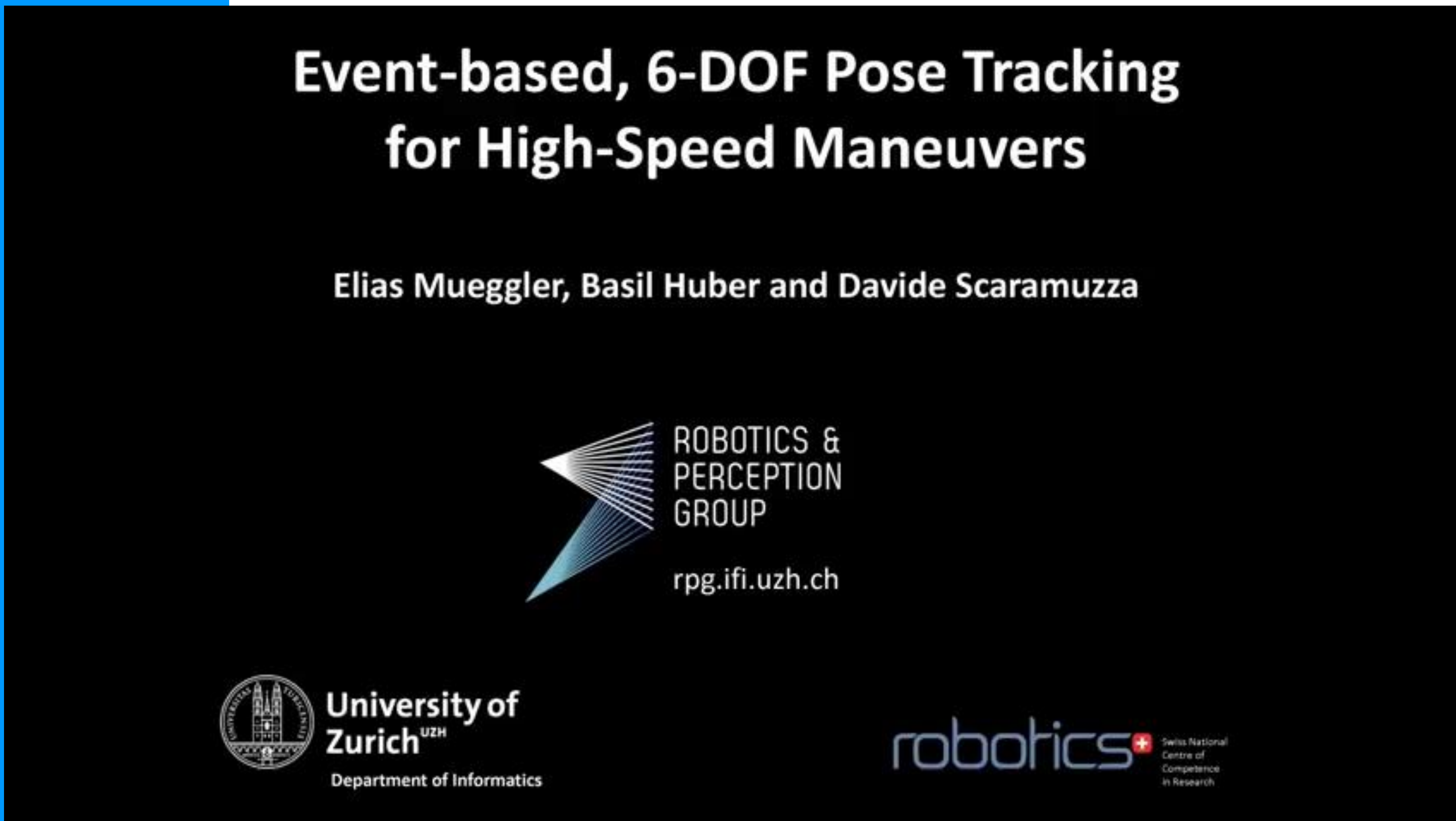
Event Cameras



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Schedule

Feb 22	Introduction
Mar 1	**Geometry, Camera Model, Calibration**
Mar 8	Features, Tracking / Matching
Mar 15	**Project Proposals by Students**
Mar 22	Structure from Motion (SfM) + papers
Mar 29	Dense Correspondence (stereo / optical flow) + papers
Apr 5	Easter break
Apr 12	Bundle Adjustment & SLAM + papers
Apr 19	**Student Midterm Presentations**
Apr 26	Multi-View Stereo & Volumetric Modeling + papers
May 3	3D Modeling with Depth Sensors + papers
May 10	Guest lecture + papers
May 17	Guest lecture + papers
May 31	**Student Project Demo Day = Final Presentations**

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Reminder

- Project presentation in 2 weeks
- Form team & decide project topic
  - By March 8nd
- Talk with supervisor, submit proposal
  - By March 15th

Next class:
Features, Tracking / Matching